Foliations over positive characteristic and irreducible components

Wodson Mendson

Université de Rennes 1 - IRMAR

1 de dezembro de 2022

- The talk is based on my PhD thesis[1] defended this year at IMPA under supervision of Jorge Vitório Pereira.

[1] **Folheações de codimensão um em característica positiva e aplicações**

Structure of the talk

- Part I: Basic notions;
- Part II: Codimension one foliations in positive characteristic;
- Part III: Irreducible components of the space of codimension one foliations on $\mathbb{P}^3_{\mathbb{C}}$.

Foliations on algebraic varieties

k = algebraically closed field of characteristic $p \geq 0$ (example: $\mathbb{C}$,$\overline{\mathbb{F}}_p$)

Foliations on algebraic varieties

k = algebraically closed field of characteristic $p \geq 0$ (example: $\mathbb{C}$,$\overline{\mathbb{F}}_p$)

Definition

Let X be a smooth algebraic variety of dimension at least two defined over k*. A* ***foliation*** $\mathcal{F}$ ***of codimension*** q *on* X *consists in a coherent subsheaf* $T_{\mathcal{F}} \subset T_X$ *of rank* $\dim X - q$ *that satisfies the following properties:*

Foliations on algebraic varieties

k = algebraically closed field of characteristic $p \geq 0$ (example: $\mathbb{C}$,$\overline{\mathbb{F}}_p$)

Definition

Let X be a smooth algebraic variety of dimension at least two defined over k*. A* ***foliation*** $\mathcal{F}$ ***of codimension*** q *on* X *consists in a coherent subsheaf* $T_{\mathcal{F}} \subset T_X$ *of rank* $\dim X - q$ *that satisfies the following properties:*

- $T_{\mathcal{F}}$ *is closed under the Lie bracket,*

Foliations on algebraic varieties

k = algebraically closed field of characteristic $p \geq 0$ (example: $\mathbb{C}$,$\overline{\mathbb{F}}_p$)

Definition

Let X be a smooth algebraic variety of dimension at least two defined over k*. A* ***foliation*** $\mathcal{F}$ ***of codimension*** q *on* X *consists in a coherent subsheaf* $T_{\mathcal{F}} \subset T_X$ *of rank* $\dim X - q$ *that satisfies the following properties:*

- $T_{\mathcal{F}}$ *is closed under the Lie bracket,*
- $T_{\mathcal{F}}$ *is saturated on* T_X*, i.e, the quotient* $T_X/T_{\mathcal{F}}$ *is free torsion.*

Foliations on algebraic varieties

k = algebraically closed field of characteristic $p \geq 0$ (example: $\mathbb{C}$,$\overline{\mathbb{F}}_p$)

Definition

Let X be a smooth algebraic variety of dimension at least two defined over k*. A* ***foliation*** $\mathcal{F}$ ***of codimension*** q *on X consists in a coherent subsheaf $T_{\mathcal{F}} \subset T_X$ of rank* $\dim X - q$ *that satisfies the following properties:*

- *$T_{\mathcal{F}}$ is closed under the Lie bracket,*
- *$T_{\mathcal{F}}$ is saturated on T_X, i.e, the quotient $T_X/T_{\mathcal{F}}$ is free torsion.*

The singular set of $\mathcal{F}$ is defined by

$$\text{sing}(\mathcal{F}) = \{x \in X \mid (T_X/T_{\mathcal{F}})_x \text{is not a free } \mathcal{O}_{X,x}\text{-module}\}.$$

Codimension one foliations ($q = 1$)

Let $\mathcal{F}$ be a codimension one foliation on X.

- **normal sheaf** of $\mathcal{F}$:

$$N_{\mathcal{F}} = (T_X/T_{\mathcal{F}})^{**}$$

- **conormal sheaf** of $\mathcal{F}$:

$$\Omega^1_{X/\mathcal{F}} = \{\omega \in \Omega^1_{X/\mathrm{k}} \mid i_v\omega = 0 \quad \forall v \in T_{\mathcal{F}}\} \cong N^*_{\mathcal{F}}$$

Codimension one foliations ($q = 1$)

Let $\mathcal{F}$ be a codimension one foliation on X.

- **normal sheaf** of $\mathcal{F}$:
$$N_{\mathcal{F}} = (T_X/T_{\mathcal{F}})^{**}$$
- **conormal sheaf** of $\mathcal{F}$:
$$\Omega^1_{X/\mathcal{F}} = \{\omega \in \Omega^1_{X/\,\mathrm{k}} \mid i_v\omega = 0 \quad \forall v \in T_{\mathcal{F}}\} \cong N^*_{\mathcal{F}}$$

The inclusion $N^*_{\mathcal{F}} \subset \Omega^1_{X/\,\mathrm{k}}$ determines a global section

$$0 \neq \omega \in \mathrm{H}^0(X, \Omega^1_{X/\,\mathrm{k}} \otimes N_{\mathcal{F}})$$

with zeros of codimension ≥ 2. Since $T_{\mathcal{F}}$ is stable by the Lie bracket we have the integrability condition: $\omega \wedge d\omega = 0$.

Codimension one foliations ($q = 1$)

Let $\mathcal{F}$ be a codimension one foliation on X.

- **normal sheaf** of $\mathcal{F}$:

$$N_{\mathcal{F}} = (T_X / T_{\mathcal{F}})^{**}$$

- **conormal sheaf** of $\mathcal{F}$:

$$\Omega^1_{X/\mathcal{F}} = \{\omega \in \Omega^1_{X/\,\mathrm{k}} \mid i_v \omega = 0 \quad \forall v \in T_{\mathcal{F}}\} \cong N^*_{\mathcal{F}}$$

The inclusion $N^*_{\mathcal{F}} \subset \Omega^1_{X/\,\mathrm{k}}$ determines a global section

$$0 \neq \omega \in \mathrm{H}^0(X, \Omega^1_{X/\,\mathrm{k}} \otimes N_{\mathcal{F}})$$

with zeros of codimension ≥ 2. Since $T_{\mathcal{F}}$ is stable by the Lie bracket we have the integrability condition: $\omega \wedge d\omega = 0$.

Reciprocally, if ω is a global section of $\Omega^1_{X/\,\mathrm{k}} \otimes \mathcal{I}$ for some invertible sheaf $\mathcal{I}$, with zeros of codimension at least two and integrable then we get a saturated subsheaf of T_X closed by Lie bracket via the kernel of the contraction map:

$$\gamma_\omega : T_X \longrightarrow \mathcal{I}$$

Codimension one foliation: definition II

Definition

Let $\mathcal{I}$ be a invertible sheaf on X. A codimension one foliation on X with normal sheaf $\mathcal{I}$ is determined by nonzero global section of $\omega \in \mathrm{H}^0(X, \Omega^1_{X/\mathrm{k}} \otimes \mathcal{I})$ that satisfies the conditions:

- $\omega \wedge d\omega = 0$,
- $\operatorname{codim} \operatorname{sing}(\omega) \geq 2$.

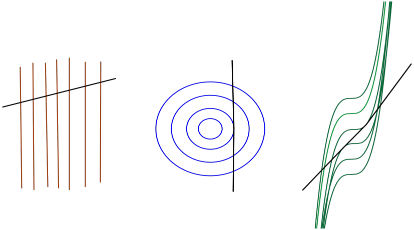
Codimension one foliation: definition II

Definition

Let $\mathcal{I}$ be a invertible sheaf on X. A codimension one foliation on X with normal sheaf $\mathcal{I}$ is determined by nonzero global section of $\omega \in \mathrm{H}^0(X, \Omega^1_{X/\mathrm{k}} \otimes \mathcal{I})$ that satisfies the conditions:

- $\omega \wedge d\omega = 0$,
- $\operatorname{codim} \operatorname{sing}(\omega) \geq 2$.

When $X = \mathbb{P}^{\mathrm{n}}_{\mathrm{k}}$ these objects are very explicit, we have the notion of **degree**: the number of tangencies of a generic line in $\mathbb{P}^{\mathrm{n}}_{\mathrm{k}}$ with the foliation.



Codimension one foliations on projective spaces

Using the Euler exact sequence for projective spaces

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0$$

we conclude that a codimension one foliation of degree d on $\mathbb{P}^n_k$ is given by a homogeneous 1-form on the affine space $\mathbb{A}^{n+1}_k$

$$\sigma = A_0 dx_0 + \cdots + A_n dx_n$$

Codimension one foliations on projective spaces

Using the Euler exact sequence for projective spaces

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0$$

we conclude that a codimension one foliation of degree d on $\mathbb{P}^n_k$ is given by a homogeneous 1-form on the affine space $\mathbb{A}^{n+1}_k$

$$\sigma = A_0 dx_0 + \cdots + A_n dx_n$$

where $A_0 \ldots, A_n \in k[x_0, \ldots, x_n]$ are homogeneous of degree $d+1$ and such that $\mathrm{sing}(\sigma) = \mathcal{Z}(A_0 \ldots, A_n)$ has codimension ≥ 2 and with σ having the following properties:

$$i_R\sigma = \sum_i A_i x_i = 0 \qquad \sigma \wedge d\sigma = 0.$$

Codimension one foliations in positive characteristic

k = algebraically closed field of characteristic $p > 0$.

R = k-domain (example: $R = \mathrm{k}[x_1, ..., x_n], \mathrm{k}[[x_1, \ldots, x_n]]$)

Codimension one foliations in positive characteristic

k = algebraically closed field of characteristic $p > 0$.

R = k-domain (example: $R = k[x_1, ..., x_n], k[[x_1, \ldots, x_n]]$)

Let v, v_1 and v_2 k-derivations of R. Properties:

- The p-iteration of v, v^p, is a k-derivation,

Codimension one foliations in positive characteristic

k = algebraically closed field of characteristic $p > 0$.

R = k-domain (example: $R = \mathrm{k}[x_1, ..., x_n], \mathrm{k}[[x_1, \ldots, x_n]]$)

Let v, v_1 and v_2 k-derivations of R. Properties:

- The p-iteration of v, v^p, is a k-derivation,
- If v_1, v_2 are k-deriations of R then

$$(v_1 + v_2)^p = v_1^p + v_2^p + \sum_{i=1}^{p-1} s_i(v_1, v_2)$$

with $s_i(v_1, v_2)$ is in the Lie algebra generate by v_1, v_2,

Codimension one foliations in positive characteristic

k = algebraically closed field of characteristic $p > 0$.

R = k-domain (example: $R = \mathrm{k}[x_1, ..., x_n], \mathrm{k}[[x_1, \ldots, x_n]]$)

Let v, v_1 and v_2 k-derivations of R. Properties:

- The p-iteration of v, v^p, is a k-derivation,
- If v_1, v_2 are k-deriations of R then

$$(v_1 + v_2)^p = v_1^p + v_2^p + \sum_{i=1}^{p-1} s_i(v_1, v_2)$$

  with $s_i(v_1, v_2)$ is in the Lie algebra generate by v_1, v_2,
- For any $f \in R$ we have

$$(fv)^p = f^p v^p - f v^{p-1}(f) v.$$

p-closed foliation

Let $\mathcal{F}$ be a foliation on a smooth algebraic variety X defined over k.

Definition

*We say that $\mathcal{F}$ is **p-closed** if $T_{\mathcal{F}}$ is closed under the p-powers.*

[2] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

p-closed foliation

Let $\mathcal{F}$ be a foliation on a smooth algebraic variety X defined over k.

Definition

*We say that $\mathcal{F}$ is **p-closed** if $T_{\mathcal{F}}$ is closed under the p-powers.*

The p-closed foliations are the version in positive characteristic of the class of holomorphic foliations that has meromorphic first integral. In particular:[2]

[2] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

p-closed foliation

Let $\mathcal{F}$ be a foliation on a smooth algebraic variety X defined over k.

Definition

*We say that $\mathcal{F}$ is p-**closed** if $T_{\mathcal{F}}$ is closed under the p-powers.*

The p-closed foliations are the version in positive characteristic of the class of holomorphic foliations that has meromorphic first integral. In particular:[2]

Theorem (Brunella-Nicolau)

Let X be a smooth projective variety over k and $\mathcal{F}$ be a codimension one foliation. Then, $\mathcal{F}$ is p-closed if and only if there are infinitely many $\mathcal{F}$-invariant hypersurfaces.

[2] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

Example

Example

Let k *be an algebraically closed field of characteristic* $p > 0$ *and* $\mathcal{F}$ *the foliation on* $\mathbb{A}^2_{\mathrm{k}}$ *defined by the* 1*-form*

$$\omega = ydx - \alpha xdy$$

for some $\alpha \in \mathrm{k}^*$*. Then,* $\mathcal{F}$ *is* p*-closed if and olny if* $\alpha \in \mathbb{F}_p$*.*

First, note that a vector field v is tangent to $\mathcal{F}$ if and only if $v = g \cdot v_1$ for some $g \in \mathrm{k}[x, y]$ where $v_1 = \alpha x\partial_x + y\partial_y$, and $v_1^p = \alpha^p x\partial_x + y\partial_y$ is tangent to $\mathcal{F}$ is and only if $\alpha \in \mathbb{F}_p$.

Despite some analogies, some objects behave differently.[3]

[3] J.V.Pereira - **Invariant hypersurfaces for positive characteristic vector fields**

Despite some analogies, some objects behave differently.[3]

Proposition (J.V.Pereira)

Let $\mathcal{F}$ be a foliation $\mathbb{P}^2_{\mathrm{k}}$ and suppose that $\deg(\mathcal{F}) < p - 1$. Then, $\mathcal{F}$ has an invariant algebraic curve.

By a result of Jouanolou for foliations defined over $\mathbb{C}$ the picture is totally different.

[3] J.V.Pereira - **Invariant hypersurfaces for positive characteristic vector fields**

Despite some analogies, some objects behave differently.[3]

Proposition (J.V.Pereira)

Let $\mathcal{F}$ be a foliation $\mathbb{P}^2_{\mathrm{k}}$ and suppose that $\deg(\mathcal{F}) < p - 1$. Then, $\mathcal{F}$ has an invariant algebraic curve.

By a result of Jouanolou for foliations defined over $\mathbb{C}$ the picture is totally different.

Theorem

For every $d \in \mathbb{Z}_{>1}$ the foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by the vector field

$$v_d = (xy^d - 1)\frac{\partial}{\partial x} - (x^d - y^{d+1})\frac{\partial}{\partial y}$$

has no algebraic solutions.

[3] J.V.Pereira - **Invariant hypersurfaces for positive characteristic vector fields**

The Cartier Operator

- k = algebraically closed field of characteristic $p > 0$
- R = local regular k-domain which is localization of a k-domain of finite type (example: $\mathcal{O}_{X,x}$)
- $t_1, \ldots, t_r$ = a regular system of parameters of R.

The Cartier Operator

- k = algebraically closed field of characteristic $p > 0$
- R = local regular k-domain which is localization of a k-domain of finite type (example: $\mathcal{O}_{X,x}$)
- $t_1, \ldots, t_r$ = a regular system of parameters of R.

From $\{t_1, \ldots, t_t\}$ we get $\{dt_1, \ldots, dt_r\}$ a basis for $\Omega^1_{R/\mathrm{k}}$. The ring R is a free R^p-module with base given by all monomials of type $t_1^{a_1} \cdots t_r^{a_r}$ with $0 \leq a_i \leq p-1$ for all i.

The Cartier Operator

- k = algebraically closed field of characteristic $p > 0$
- R = local regular k-domain which is localization of a k-domain of finite type (example: $\mathcal{O}_{X,x}$)
- $t_1, \ldots, t_r$ = a regular system of parameters of R.

From $\{t_1, \ldots, t_t\}$ we get $\{dt_1, \ldots, dt_r\}$ a basis for $\Omega^1_{R/\,\mathrm{k}}$. The ring R is a free R^p-module with base given by all monomials of type $t_1^{a_1} \cdots t_r^{a_r}$ with $0 \leq a_i \leq p-1$ for all i.

- **closed 1-forms:**

$$Z^1_{R/\,\mathrm{k}} = \{\omega \in \Omega^1_{R/\,\mathrm{k}} \mid d\omega = 0\}$$

The Cartier Operator

- k = algebraically closed field of characteristic $p > 0$
- R = local regular k-domain which is localization of a k-domain of finite type (example: $\mathcal{O}_{X,x}$)
- $t_1, \ldots, t_r$ = a regular system of parameters of R.

From $\{t_1, \ldots, t_t\}$ we get $\{dt_1, \ldots, dt_r\}$ a basis for $\Omega^1_{R/\,\mathrm{k}}$. The ring R is a free R^p-module with base given by all monomials of type $t_1^{a_1} \cdots t_r^{a_r}$ with $0 \leq a_i \leq p-1$ for all i.

- **closed 1-forms:**

$$Z^1_{R/\,\mathrm{k}} = \{\omega \in \Omega^1_{R/\,\mathrm{k}} \mid d\omega = 0\}$$

- **exact 1-forms:**

$$B^1_{R/\,\mathrm{k}} = \{\omega \in \Omega^1_{R/\,\mathrm{k}} \mid \omega = dg\}$$

The Cartier Operator

- k = algebraically closed field of characteristic $p > 0$
- R = local regular k-domain which is localization of a k-domain of finite type (example: $\mathcal{O}_{X,x}$)
- $t_1, \ldots, t_r$ = a regular system of parameters of R.

From $\{t_1, \ldots, t_t\}$ we get $\{dt_1, \ldots, dt_r\}$ a basis for $\Omega^1_{R/\,\mathrm{k}}$. The ring R is a free R^p-module with base given by all monomials of type $t_1^{a_1} \cdots t_r^{a_r}$ with $0 \leq a_i \leq p-1$ for all i.

- **closed 1-forms:**

$$Z^1_{R/\,\mathrm{k}} = \{\omega \in \Omega^1_{R/\,\mathrm{k}} \mid d\omega = 0\}$$

- **exact 1-forms:**

$$B^1_{R/\,\mathrm{k}} = \{\omega \in \Omega^1_{R/\,\mathrm{k}} \mid \omega = dg\}$$

- **obstruction:**

$$H^1_{R/\,\mathrm{k}} = Z^1_{R/\,\mathrm{k}} / B^1_{R/\,\mathrm{k}}$$

Cartier Operator

Consider the R^p-module

$$M(t_1, \dots, t_r) = R^p t_1^{p-1} dt_1 \oplus \cdots \oplus R^p t_r^{p-1} dt_r$$

Proposition

Every element $\sigma \in Z^1_{R/\mathrm{k}}$ can be written uniquely as $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in B^1_{R/\mathrm{k}}$ and $\sigma_2 \in M(t_1, \dots, t_r)$.

Cartier Operator

Consider the R^p-module

$$M(t_1, \dots, t_r) = R^p t_1^{p-1} dt_1 \oplus \cdots \oplus R^p t_r^{p-1} dt_r$$

Proposition

Every element $\sigma \in Z^1_{R/\,\mathrm{k}}$ can be written uniquely as $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in B^1_{R/\,\mathrm{k}}$ and $\sigma_2 \in M(t_1, \dots, t_r)$.

The **Cartier Operator** is the map

$$\mathbf{C}\colon Z^1_{R/\,\mathrm{k}} \longrightarrow \Omega^1_{R/\,\mathrm{k}}$$

$$dg + \sum_{i=1}^{r} u_i^p t_i^{p-1} dt_i \mapsto \sum_{i=1}^{r} u_i dt_i$$

Fundamental formula

The **Cartier Operator** can be defined in more intrinsic terms as the inverse of the isomorphism[4]

$$\gamma\colon \Omega^1_{R/\,k} \longrightarrow Z^1_{R/\,k} \longrightarrow H^1_{R/\,k}$$
$$adt \mapsto a^p t^{p-1} dt \mapsto [a^p t^{p-1} dt].$$

[4] Michel Brion, Shrawan Kumar - **Frobenius splitting methods in geometry and representation theory**, 1.3.4 Theorem

Fundamental formula

The **Cartier Operator** can be defined in more intrinsic terms as the inverse of the isomorphism[4]

$$\gamma \colon \Omega^1_{R/\,\mathrm{k}} \longrightarrow Z^1_{R/\,\mathrm{k}} \longrightarrow H^1_{R/\,\mathrm{k}}$$
$$adt \mapsto a^p t^{p-1} dt \mapsto [a^p t^{p-1} dt].$$

Theorem

Let $\omega \in \Omega^1_{R/\,\mathrm{k}}$ be a closed 1-form and $v \in \mathrm{Der}_{\mathrm{k}}(R)$ be a derivation. Then,

$$i_v\, \boldsymbol{C}(\omega)^p = i_{v^p}\omega - v^{p-1}(i_v\omega).$$

[4] Michel Brion, Shrawan Kumar - **Frobenius splitting methods in geometry and representation theory**, 1.3.4 Theorem

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over* k *and denote by* $\mathcal{Z}^1_{X/\mathrm{k}}$ *the subsheaf of* $\Omega^1_{X/\mathrm{k}}$ *which consists of closed* 1*-forms. There exists a operator, the* ***Cartier Operator***, $\boldsymbol{C} : \mathcal{Z}^1_{X/\mathrm{k}} \longrightarrow \Omega^1_{X/\mathrm{k}}$ *determined by the following properties:*

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over* k *and denote by* $\mathcal{Z}^1_{X/\mathrm{k}}$ *the subsheaf of* $\Omega^1_{X/\mathrm{k}}$ *which consists of closed* 1*-forms. There exists a operator, the* ***Cartier Operator***, $\boldsymbol{C} : \mathcal{Z}^1_{X/\mathrm{k}} \longrightarrow \Omega^1_{X/\mathrm{k}}$ *determined by the following properties:*

1. $\boldsymbol{C}(\sigma_1 + \sigma_2) = \boldsymbol{C}(\sigma_1) + \boldsymbol{C}(\sigma_2)$,

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over k and denote by $\mathcal{Z}^1_{X/\mathrm{k}}$ the subsheaf of $\Omega^1_{X/\mathrm{k}}$ which consists of closed 1-forms. There exists a operator, the* ***Cartier Operator***, $\boldsymbol{C} : \mathcal{Z}^1_{X/\mathrm{k}} \longrightarrow \Omega^1_{X/\mathrm{k}}$ *determined by the following properties:*

1. $\boldsymbol{C}(\sigma_1 + \sigma_2) = \boldsymbol{C}(\sigma_1) + \boldsymbol{C}(\sigma_2)$,
2. $\boldsymbol{C}(f^p \sigma_1) = f\, \boldsymbol{C}(\sigma_1)$,

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over k and denote by $\mathcal{Z}^1_{X/\,\mathrm{k}}$ the subsheaf of $\Omega^1_{X/\,\mathrm{k}}$ which consists of closed 1-forms. There exists a operator, the* ***Cartier Operator***, $\boldsymbol{C} : \mathcal{Z}^1_{X/\,\mathrm{k}} \longrightarrow \Omega^1_{X/\,\mathrm{k}}$ *determined by the following properties:*

1. $\boldsymbol{C}(\sigma_1 + \sigma_2) = \boldsymbol{C}(\sigma_1) + \boldsymbol{C}(\sigma_2)$,
2. $\boldsymbol{C}(f^p \sigma_1) = f\, \boldsymbol{C}(\sigma_1)$,
3. $\boldsymbol{C}(df) = 0$,

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over k and denote by $\mathcal{Z}^1_{X/\,\mathrm{k}}$ the subsheaf of $\Omega^1_{X/\,\mathrm{k}}$ which consists of closed 1-forms. There exists a operator, the **Cartier Operator**, $\boldsymbol{C} : \mathcal{Z}^1_{X/\,\mathrm{k}} \longrightarrow \Omega^1_{X/\,\mathrm{k}}$ determined by the following properties:*

1. $\boldsymbol{C}(\sigma_1 + \sigma_2) = \boldsymbol{C}(\sigma_1) + \boldsymbol{C}(\sigma_2)$,
2. $\boldsymbol{C}(f^p \sigma_1) = f\,\boldsymbol{C}(\sigma_1)$,
3. $\boldsymbol{C}(df) = 0$,
4. $\boldsymbol{C}(f^{p-1} df) = df$,

Some properties

Proposition

[a] *Let X be a smooth algebraic variety defined over* k *and denote by* $\mathcal{Z}^1_{X/\mathrm{k}}$ *the subsheaf of* $\Omega^1_{X/\mathrm{k}}$ *which consists of closed* 1*-forms. There exists a operator, the* ***Cartier Operator***, $\boldsymbol{C} : \mathcal{Z}^1_{X/\mathrm{k}} \longrightarrow \Omega^1_{X/\mathrm{k}}$ *determined by the following properties:*

- i. $\boldsymbol{C}(\sigma_1 + \sigma_2) = \boldsymbol{C}(\sigma_1) + \boldsymbol{C}(\sigma_2)$,
- ii. $\boldsymbol{C}(f^p \sigma_1) = f\, \boldsymbol{C}(\sigma_1)$,
- iii. $\boldsymbol{C}(df) = 0$,
- iv. $\boldsymbol{C}(f^{p-1} df) = df$,
- v. $\boldsymbol{C}(\frac{df}{f}) = \frac{df}{f}$

for any local sections $f \in \mathcal{O}_X$, $\sigma_1, \sigma_2 \in \mathcal{Z}^1_{X/\mathrm{k}}$.

[a] Seshadr - **L'opération de Cartier**

The non-p-closed foliations and the p-distribution

- X = smooth algebraic variety of dimension ≥ 2 defined over k
- $\mathcal{F}$ = codimension one foliation non-p-closed on X

The non-p-closed foliations and the p-distribution

- X = smooth algebraic variety of dimension ≥ 2 defined over k
- $\mathcal{F}$ = codimension one foliation non-p-closed on X

Theorem (D. Cerveau, A. Lins Neto, F. Loray, J.V. Pereira, F. Touzet)

Let ω be a rational 1-form. Suppose that ω is integrable and that v is a rational vector field such that $i_v\omega = 0$. If $f = i_{v_p}\omega \neq 0$ then $d(f^{p-1}\omega) = 0$.[a]

[a] **Complex codimension one singular foliations and Godbillon-Vey sequences**

The non-p-closed foliations and the p-distribution

- X = smooth algebraic variety of dimension ≥ 2 defined over k
- $\mathcal{F}$ = codimension one foliation non-p-closed on X

Theorem (D. Cerveau, A. Lins Neto, F. Loray, J.V. Pereira, F. Touzet)

Let ω be a rational 1-form. Suppose that ω is integrable and that v is a rational vector field such that $i_v\omega = 0$. If $f = i_{v_p}\omega \neq 0$ then $d(f^{p-1}\omega) = 0$.[a]

[a] **Complex codimension one singular foliations and Godbillon-Vey sequences**

Let ω be a closed 1-form defining $\mathcal{F}$. Consider the subsheaf $T_{\mathcal{C}_{\mathcal{F}}}$ of $T_{\mathcal{F}}$ defining by

$$T_{\mathcal{C}_{\mathcal{F}}} = \{v \in T_{\mathcal{F}} \mid i_v\,\mathbf{C}(\omega) = 0\} \tag{1}$$

where $\mathbf{C}$ is the Cartier Operator.

The p-curvature morphism

$$T_{\mathcal{C}_{\mathcal{F}}} = \{v \in T_{\mathcal{F}} \mid i_v \, \mathbf{C}(\omega) = 0\} \quad (2)$$

By the Cartier Operator properties it follows that $T_{\mathcal{C}_{\mathcal{F}}}$ is independent of the closed 1-form defining $\mathcal{F}$ and is a saturated subsheaf of T_X.

The p-curvature morphism

$$T_{\mathcal{C}_{\mathcal{F}}} = \{v \in T_{\mathcal{F}} \mid i_v\, \mathbf{C}(\omega) = 0\} \tag{2}$$

By the Cartier Operator properties it follows that $T_{\mathcal{C}_{\mathcal{F}}}$ is independent of the closed 1-form defining $\mathcal{F}$ and is a saturated subsheaf of T_X.

Definition

*Let $\mathcal{F}$ be a codimension one foliation non-p-closed on X. The p-**distribution** associated to $\mathcal{F}$ is the distribution defined by the sheaf $T_{\mathcal{C}_{\mathcal{F}}}$.*

The p-curvature morphism

$$T_{\mathcal{C}_{\mathcal{F}}} = \{v \in T_{\mathcal{F}} \mid i_v\, \mathbf{C}(\omega) = 0\} \tag{2}$$

By the Cartier Operator properties it follows that $T_{\mathcal{C}_{\mathcal{F}}}$ is independent of the closed 1-form defining $\mathcal{F}$ and is a saturated subsheaf of T_X.

Definition

*Let $\mathcal{F}$ be a codimension one foliation non-p-closed on X. The p-**distribution** associated to $\mathcal{F}$ is the distribution defined by the sheaf $T_{\mathcal{C}_{\mathcal{F}}}$.*

Example

The fundamental formula implies that if $\dim X = 2$ then $T_{\mathcal{C}_{\mathcal{F}}}$ is the null sheaf. Indeed, given $v \in T_{\mathcal{F}}$ we have $0 \neq i_{v^p}\omega = i_v\, \boldsymbol{C}(\omega)^p$.

The p-curvature morphism

$$T_{\mathcal{C}_{\mathcal{F}}} = \{v \in T_{\mathcal{F}} \mid i_v\, \mathbf{C}(\omega) = 0\} \tag{2}$$

By the Cartier Operator properties it follows that $T_{\mathcal{C}_{\mathcal{F}}}$ is independent of the closed 1-form defining $\mathcal{F}$ and is a saturated subsheaf of T_X.

Definition

*Let $\mathcal{F}$ be a codimension one foliation non-p-closed on X. The p-**distribution** associated to $\mathcal{F}$ is the distribution defined by the sheaf $T_{\mathcal{C}_{\mathcal{F}}}$.*

Example

The fundamental formula implies that if $\dim X = 2$ then $T_{\mathcal{C}_{\mathcal{F}}}$ is the null sheaf. Indeed, given $v \in T_{\mathcal{F}}$ we have $0 \neq i_{v^p}\omega = i_v\, \boldsymbol{C}(\omega)^p$.

Consider the following morphism of sets sheafs

$$\psi_{\mathcal{F}} \colon T_{\mathcal{F}} \longrightarrow \frac{T_X}{T_{\mathcal{F}}}$$

which associates $v \mapsto v^p \mod T_{\mathcal{F}}$.

The p-curvature morphism and Frobenius

The properties of derivations over positive characteristic implies that $\psi_{\mathcal{F}}$ is a group morphism.

The p-curvature morphism and Frobenius

The properties of derivations over positive characteristic implies that $\psi_{\mathcal{F}}$ is a group morphism.

Definition

The ***p-curvature morphism associated to*** $\mathcal{F}$ *is the* $\mathcal{O}_X$*-morphism:*

$$\varphi_{\mathcal{F}} \colon F_X^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$

$$\sum_i f_i \otimes v_i \mapsto \sum_i f_i v_i^p.$$

The p-curvature morphism and Frobenius

The properties of derivations over positive characteristic implies that $\psi_{\mathcal{F}}$ is a group morphism.

Definition

*The **p-curvature morphism associated to** $\mathcal{F}$ is the $\mathcal{O}_X$-morphism:*

$$\varphi_{\mathcal{F}}\colon F_X^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$
$$\sum_i f_i \otimes v_i \mapsto \sum_i f_i v_i^p.$$

In the conditions above, the foliation $\mathcal{F}$ is p-closed if and only if $\varphi_{\mathcal{F}} \equiv 0$.

The p-curvature morphism and Frobenius

The properties of derivations over positive characteristic implies that $\psi_{\mathcal{F}}$ is a group morphism.

Definition

The ***p-curvature morphism associated to $\mathcal{F}$*** *is the $\mathcal{O}_X$-morphism:*

$$\varphi_{\mathcal{F}} \colon F_X^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$
$$\sum_i f_i \otimes v_i \mapsto \sum_i f_i v_i^p.$$

In the conditions above, the foliation $\mathcal{F}$ is p-closed if and only if $\varphi_{\mathcal{F}} \equiv 0$.

Recall: The **absolute Frobenius** morphism, denoted by F_X, consists in the morphism that is the identity on topological spaces and is the p-power on functions

$$F_X = (f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (X, \mathcal{O}_X)$$

where $f = id$ and $f^{\#} \colon a \mapsto a^p$.

The p-curvature

Consider the p-curvature morphism

$$\varphi_{\mathcal{F}} \colon F_X^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$
$$\sum_i f_i \otimes v_i \mapsto \sum_i f_i v_i^p.$$

The p-curvature

Consider the p-curvature morphism

$$\begin{aligned} \varphi_{\mathcal{F}} \colon F_X^* T_{\mathcal{F}} &\longrightarrow N_{\mathcal{F}} \\ \sum_i f_i \otimes v_i &\mapsto \sum_i f_i v_i^p. \end{aligned}$$

Proposition

We have $\mathrm{Ker}(\varphi_{\mathcal{F}}) = F_X^* T_{\mathcal{C}_{\mathcal{F}}}$ *where* F_X *is the absolute Frobenius morphism and there exists a effective divisor* $\Delta_{\mathcal{F}} \in \mathrm{Div}(X)$ *such that the sequence*

$$0 \longrightarrow F_X^* T_{\mathcal{C}_{\mathcal{F}}} \longrightarrow F_X^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\Delta_{\mathcal{F}}) \longrightarrow 0$$

is exact in codimension one, i.e, exact outside a closed set of codimension ≥ 2.

The p-distribution and the p-divisor

Definition

*Let $\mathcal{F}$ be a foliation that is not p-closed on X. The **p-distribution** associated to $\mathcal{F}$ is the subsheaf of T_X defined by $T_{\mathcal{C}_{\mathcal{F}}}$. The **p-divisor** of $\mathcal{F}$ is the divisor $\Delta_{\mathcal{F}}$.*

An interesting property of the p-divisor consists of the following proposition.

The p-distribution and the p-divisor

Definition

Let $\mathcal{F}$ be a foliation that is not p-closed on X. The ***p-distribution*** *associated to $\mathcal{F}$ is the subsheaf of T_X defined by $T_{\mathcal{C}_{\mathcal{F}}}$. The* ***p-divisor*** *of $\mathcal{F}$ is the divisor $\Delta_{\mathcal{F}}$.*

An interesting property of the p-divisor consists of the following proposition.

Proposition

Let X be a smooth variety over k *and $\mathcal{F}$ be a foliation on X that is not p-closed. Let H be an irreducible hypersurface on X. If H is $\mathcal{F}$-invariant then $\mathrm{ord}_H(\Delta_{\mathcal{F}}) > 0$. Reciprocally, if $\mathrm{ord}_H(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$ then H is $\mathcal{F}$-invariant.*

Some consequences

Proposition

Let $\mathcal{F}$ be a codimension one foliation on a smooth projective variety X of dimension ≥ 2 defined over k*. Suppose that $\mathcal{F}$ is not p-closed. Then,*

$$\mathcal{O}_X(\Delta_{\mathcal{F}}) = \omega_{\mathcal{F}}^{\otimes p} \otimes (\omega^*_{\mathcal{C}_{\mathcal{F}}})^{\otimes p} \otimes N_{\mathcal{F}}$$

in the group $\mathrm{Pic}(X)$.

Some consequences

Proposition

Let $\mathcal{F}$ be a codimension one foliation on a smooth projective variety X of dimension ≥ 2 defined over k*. Suppose that $\mathcal{F}$ is not p-closed. Then,*

$$\mathcal{O}_X(\Delta_{\mathcal{F}}) = \omega_{\mathcal{F}}^{\otimes p} \otimes (\omega^*_{\mathcal{C}_{\mathcal{F}}})^{\otimes p} \otimes N_{\mathcal{F}}$$

in the group $\mathrm{Pic}(X)$.

When $X = \mathbb{P}^{\mathrm{n}}_{\mathrm{k}}$ the proposition above implies the following **degree formula:**

$$\deg(\Delta_{\mathcal{F}}) = p(d - \deg(\mathcal{C}_{\mathcal{F}}) - 1) + d + 2 \tag{3}$$

The p-divisor and properties

Proposition

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ such that $p \nmid \deg(N_{\mathcal{F}})$. Then, $\mathcal{F}$ admits an invariant hypersurface.

The p-divisor and properties

Proposition

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ such that $p \nmid \deg(N_{\mathcal{F}})$. Then, $\mathcal{F}$ admits an invariant hypersurface.

Demonstração.

If $\mathcal{F}$ is p-closed then $\mathcal{F}$ admits infinitely many solutions. So, we can assume that $\mathcal{F}$ is not p-closed. Since $p \nmid \deg(N_{\mathcal{F}})$, it follows from degree formula that $\deg(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$. In particular, $\Delta_{\mathcal{F}}$ is not a p-factor and there is a prime divisor H in the support of $\Delta_{\mathcal{F}}$ such that $\mathrm{ord}_H(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$. This divisor defines a $\mathcal{F}$-invariant hypersurface. □

Example - Foliations on surfaces and the p-divisor

Let X be a projective smooth surface defined over k. A foliation on X can be defined by a system $\{(U_i, \omega_i, v_i)\}_{i \in I}$ such that:

- $\{U_i\}_{i \in I}$ is a open cover of X.
- For each $i \in I$ we have $v_i \in T_X(U_i)$, $\omega_i \in \Omega^1_{X/\mathrm{k}}(U_i)$ such that $i_{v_i}\omega_i = 0$.
- In $U_i \cap U_j$ we have $\omega_i = f_{ij}\omega_j$ and $v_i = g_{ij}v_j$ for some functions $f_{ij}, g_{ij} \in \mathcal{O}^*_X(U_{ij})$.
- For each $i \in I$ we have $\mathrm{codim}(\omega_i) \geq 2$ and $\mathrm{codim}(v_i) \geq 2$.

Example - Foliations on surfaces and the p-divisor

Let X be a projective smooth surface defined over k. A foliation on X can be defined by a system $\{(U_i, \omega_i, v_i)\}_{i\in I}$ such that:

- $\{U_i\}_{i\in I}$ is a open cover of X.
- For each $i \in I$ we have $v_i \in T_X(U_i)$, $\omega_i \in \Omega^1_{X/\mathrm{k}}(U_i)$ such that $i_{v_i}\omega_i = 0$.
- In $U_i \cap U_j$ we have $\omega_i = f_{ij}\omega_j$ and $v_i = g_{ij}v_j$ for some functions $f_{ij}, g_{ij} \in \mathcal{O}^*_X(U_{ij})$.
- For each $i \in I$ we have $\mathrm{codim}(\omega_i) \geq 2$ and $\mathrm{codim}(v_i) \geq 2$.

The collection $\{f_{ij}^{-1}\}, \{g_{ij}\}$ define elements of $H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X)$ and the line bundles associated are the **conormal** $\Omega^1_{X/\mathcal{F}}$ and the **cotangent** $\Omega^1_{\mathcal{F}}$ bundles. Any divisor in the linear class of $\Omega^1_{\mathcal{F}}$ is called the **canonical divisor** of $\mathcal{F}$ and it denoted by $K_{\mathcal{F}}$.

Explicit construction of the p-divisor

Let $\mathcal{F} = \{(U_i, \omega_i, v_i)\}$ be a foliation on X that is not p-closed. In U_{ij} we have relations:

$$\omega_i = f_{ij}\omega_j \qquad v_i = g_{ij}v_j.$$

Since we are assuming that $\mathcal{F}$ is not p-closed:

$$0 \neq i_{v_i^p}\omega_i = i_{(g_{ij}v_j)^p} f_{ij}\omega_j = i_{(g_{ij}^p v_j^p + g_{ij}v_j^{p-1}(g_{ij}^{p-1})v_j)} f_{ij}\omega_j = g_{ij}^p f_{ij} i_{v_j^p}\omega_j \neq 0.$$

The $\{i_{v_i^p}\omega_i\}_{i\in I}$ defines a section $0 \neq s_{\mathcal{F}} \in \mathrm{H}^0(X, (\Omega^1_{\mathcal{F}})^{\otimes p} \otimes N_{\mathcal{F}})$.

Remark

The p-divisor associated to $\mathcal{F}$ is the zero divisor of the section $s_{\mathcal{F}}$:

$$\Delta_{\mathcal{F}} = (s_{\mathcal{F}})_0 \in \mathrm{Div}(X).$$

The p-divisor and properties: example I

Proposition

Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form

$$\Omega = Adx + Bdy + Cdz.$$

*Suppose that $A, B, C \in \mathbb{Z}[x, y, z]_{d+1}$ and let $p\mathbb{Z} \in$ **Spm**$(\mathbb{Z})$ be a maximal ideal such that $p > d + 2$. Let $\mathcal{F}_p$ be a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$ obtained by reduction modulo $p\mathbb{Z}$ of the coefficients of Ω. If $\Delta_{\mathcal{F}_p}$* ***is irreducible*** *then $\mathcal{F}$* ***has no algebraic solutions****.*

[5] Carnicer - **The Poincaré problem in the nondicritical case**

The p-divisor and properties: example I

Proposition

Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form

$$\Omega = Adx + Bdy + Cdz.$$

*Suppose that $A, B, C \in \mathbb{Z}[x, y, z]_{d+1}$ and let $p\mathbb{Z} \in$ **Spm**$(\mathbb{Z})$ be a maximal ideal such that $p > d + 2$. Let $\mathcal{F}_p$ be a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$ obtained by reduction modulo $p\mathbb{Z}$ of the coefficients of Ω. If $\Delta_{\mathcal{F}_p}$* ***is irreducible*** *then $\mathcal{F}$* ***has no algebraic solutions****.*

This can be used to give a simple proof of Jouanolou's Theorem which says that almost all foliations in the complex projective plane of degree $d \in \{2, 3\}$ have no algebraic solutions. The crucial point is the bound for the degree of algebraic solutions given by Carnicer.[5]

[5] Carnicer - **The Poincaré problem in the nondicritical case**

Example

- $\mathcal{F}_d$ on $\mathbb{P}^2_{\mathbb{C}}$:[6]

$$\omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

1. $(p,d) = (5,2)$:

$$\Delta_{\mathcal{F}_{5,2}} = [i_{v_2^5}\omega_2] = \{X^5Z^4 + X^4Y^5 + 2X^3Y^3Z^3 + Y^4Z^5 = 0\} \in \mathrm{Div}(\mathbb{P}^2_{\overline{\mathbb{F}}_5})$$

2. $(p,d) = (11,3)$:

$$\Delta_{\mathcal{F}_{11,3}} = [i_{v_3^{11}}\omega_3] = \{X^{19}Z^8 - 2X^{16}Y^4Z^7 + \cdots + 3XY^{11}Z^{15} + Y^8Z^{19} = 0\} \in \mathrm{Div}(\mathbb{P}^2_{\overline{\mathbb{F}}_{11}})$$

[6] **Singular**: https://www.singular.uni-kl.de/

The p-divisor on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$

Problem

Let X be a smooth algebraic surface. What we can say about $\Delta_{\mathcal{F}}$ for a generic foliation $\mathcal{F}$?

[7] W.Mendson - **Foliations on smooth algebraic surfaces over positive characteristic**

The p-divisor on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$

Problem

Let X be a smooth algebraic surface. What we can say about $\Delta_{\mathcal{F}}$ for a generic foliation $\mathcal{F}$?

- Is it true that almost all foliations on $\mathbb{P}^2_k$ have reduced p-divisor? Irreducible?

[7] W.Mendson - **Foliations on smooth algebraic surfaces over positive characteristic**

The p-divisor on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$

Problem

Let X be a smooth algebraic surface. What we can say about $\Delta_{\mathcal{F}}$ for a generic foliation $\mathcal{F}$?

- Is it true that almost all foliations on $\mathbb{P}^2_k$ have reduced p-divisor? Irreducible?

In this direction[7].

Theorem

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \nmid d_i$, if $d_i \neq 0$. Then,

[7] W.Mendson - **Foliations on smooth algebraic surfaces over positive characteristic**

The p-divisor on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$

Problem

Let X be a smooth algebraic surface. What we can say about $\Delta_{\mathcal{F}}$ for a generic foliation $\mathcal{F}$?

- Is it true that almost all foliations on $\mathbb{P}^2_k$ have reduced p-divisor? Irreducible?

In this direction[7].

Theorem

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \nmid d_i$, if $d_i \neq 0$. Then,

- *A generic foliation on $\mathbb{P}^2_k$ of degree $d \geq 1$ ($p \nmid d$) has reduced p-divisor, and*

[7] W.Mendson - **Foliations on smooth algebraic surfaces over positive characteristic**

The p-divisor on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$

Problem

Let X be a smooth algebraic surface. What we can say about $\Delta_{\mathcal{F}}$ for a generic foliation $\mathcal{F}$?

- Is it true that almost all foliations on $\mathbb{P}^2_k$ have reduced p-divisor? Irreducible?

In this direction[7].

Theorem

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \nmid d_i$, if $d_i \neq 0$. Then,

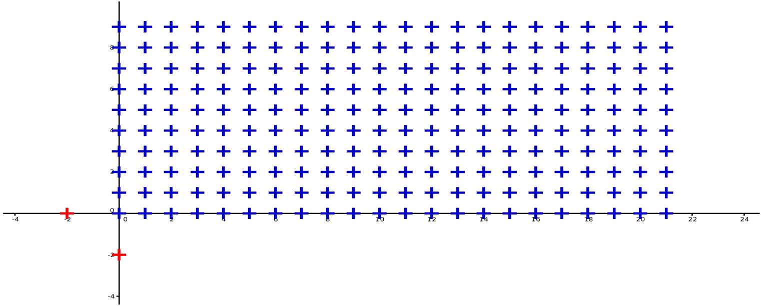
- *A generic foliation on $\mathbb{P}^2_k$ of degree $d \geq 1$ ($p \nmid d$) has reduced p-divisor, and*
- *A generic foliation on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with canonical divisor $K \equiv d_1 F + d_2 M$ has reduced p-divisor.*

[7] W.Mendson - **Foliations on smooth algebraic surfaces over positive characteristic**

Foliation of type (d_1, d_2) on $\mathbb{P}^1_k \times \mathbb{P}^1_k$

We say that a foliation on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is **of type** (d_1, d_2) if it has canonical divisor of bi-degree (d_1, d_2). The list of all possibilities is the region:[8]

$$S_0 = \{(d_1, d_2) \in \mathbb{Z}^2 \mid d_1, d_2 \geq 0\} \cup \{(-2, 0)\} \cup \{(0, -2)\}.$$



[8] Carlos Galindo, Francisco Monserrat, Jorge Olivares - **Foliations with isolated singularities on Hirzebruch surfaces**

Applications: codimension one foliation on projective spaces

A codimension one foliation of degree d on $\mathbb{P}^{\mathrm{n}}_{\mathrm{k}}$ is given by a homogeneous 1-form on $\mathbb{A}^{n+1}_{\mathrm{k}}$

$$\sigma = A_0 dx_0 + \cdots + A_n dx_n$$

where $A_0 \ldots, A_n \in \mathrm{k}[x_0, \ldots, x_n]$ are homogeneous polynomials of degree $d+1$ and such that $\mathrm{sing}(\sigma) = \mathcal{Z}(A_0 \ldots, A_n)$ has codimension ≥ 2 with σ satisfying the following conditions

$$i_R\sigma = \sum_i A_i x_i = 0 \qquad \sigma \wedge d\sigma = 0.$$

Codimension one foliations on projective spaces

The integrability condition gives equations:

$$A_i\left(\frac{\partial A_l}{\partial x_j}-\frac{\partial A_j}{\partial x_l}\right)+A_j\left(\frac{\partial A_i}{\partial x_l}-\frac{\partial A_l}{\partial x_i}\right)+A_l\left(\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}\right)=0$$

for $0\leq i<j<l\leq n$.

Codimension one foliations on projective spaces

The integrability condition gives equations:

$$A_i\left(\frac{\partial A_l}{\partial x_j}-\frac{\partial A_j}{\partial x_l}\right)+A_j\left(\frac{\partial A_i}{\partial x_l}-\frac{\partial A_l}{\partial x_i}\right)+A_l\left(\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}\right)=0$$

for $0\leq i<j<l\leq n$.

The space of codimension one foliations of degree $d\geq 0$ on $\mathbb{P}^n_k$ ($n\geq 2$) is denoted by

$$\mathbb{F}\mathrm{ol}_d(\mathbb{P}^n_k)=\{[\omega]\in\mathbb{P}(\mathrm{H}^0(\mathbb{P}^n_k,\Omega^1_{\mathbb{P}^n_k}\otimes\mathcal{O}_{\mathbb{P}^n_k}(d+2)))\mid \omega\wedge d\omega=0 \text{ and } \operatorname{codim}\operatorname{sing}(\omega)\geq 2\}$$

Problem

Describe the irreducible components of $\mathbb{F}\mathrm{ol}_d(\mathbb{P}^n_{\mathbb{C}})$.

Some components $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})(n \geq 3)$

- **Degree 0 and 1**: $\mathbb{F}\mathrm{ol}_0(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ is irreducible and for $d = 1$ the space $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has two irreducible components[9].

[9] Alcides Lins Neto - **Componentes irredutíveis dos espaços de folheações**

[10] **Irreducible components of the space of holomorphic foliations of degree two in C**$\mathrm{P}(n)$

[11] **Codimension one foliations of degree three on projective spaces**

Some components $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})(n \geq 3)$

- **Degree 0 and 1**: $\mathbb{F}\mathrm{ol}_0(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ is irreducible and for $d = 1$ the space $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has two irreducible components[9].

- **Degree 2**: For codimension one foliations with degree $d = 2$ on $\mathbb{P}^{\mathrm{n}}_{\mathbb{C}}$ Cerveau and Alcides Lins Neto showed[10] that $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has precisely six irreducible components and they describe explicitly those components.

[9] Alcides Lins Neto - **Componentes irredutíveis dos espaços de folheações**

[10] **Irreducible components of the space of holomorphic foliations of degree two in C**$\mathrm{P}(n)$

[11] **Codimension one foliations of degree three on projective spaces**

Some components $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})(n \geq 3)$

- **Degree 0 and 1**: $\mathbb{F}\mathrm{ol}_0(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ is irreducible and for $d = 1$ the space $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has two irreducible components[9].
- **Degree 2**: For codimension one foliations with degree $d = 2$ on $\mathbb{P}^{\mathrm{n}}_{\mathbb{C}}$ Cerveau and Alcides Lins Neto showed[10] that $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has precisely six irreducible components and they describe explicitly those components.
- **Degree 3**: In a recent work[11], R.C. da Costa, R. Lizarbe and J.V Pereira, using a structure theorem for codimension one foliations of degree $d = 3$ on $\mathbb{P}^{\mathrm{n}}_{\mathbb{C}}$ describe precisely 18 irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ whose the generic element has no meromorphic first integral. The authors show that $\mathbb{F}\mathrm{ol}_3(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$ has at least 24 irreducible components.

[9] Alcides Lins Neto - **Componentes irredutíveis dos espaços de folheações**
[10] **Irreducible components of the space of holomorphic foliations of degree two in** $\mathbf{C}\mathrm{P}(n)$
[11] **Codimension one foliations of degree three on projective spaces**

Components of degree $d \geq 3$

- **Rational components**: Let F, G be irreducible homogeneous polynomials of degree p and q respectively. Suppose that F and G are coprime and that $d = p + q - 2$. Then, $\omega = qFdG - pGdF$ defines a foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree d. Denote by $\mathrm{Rat}(p, q)$ the set of foliations of this type. Then, the closure $\overline{\mathrm{Rat}(p, q)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$[12].

[12] Gómez-Mont, Lins Neto - **Structural stability of foliations with a meromorphic first integral**

[13] Cerveau, Lins Neto and Edixhoven - **Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$**

Components of degree $d \geq 3$

- **Rational components**: Let F, G be irreducible homogeneous polynomials of degree p and q respectively. Suppose that F and G are coprime and that $d = p + q - 2$. Then, $\omega = qFdG - pGdF$ defines a foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree d. Denote by $\mathrm{Rat}(p, q)$ the set of foliations of this type. Then, the closure $\overline{\mathrm{Rat}(p, q)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_d(\mathbb{P}^n_{\mathbb{C}})$[12].

- **Pullback components**: Let $\mathcal{G}$ be a codimension one foliation of degree e on $\mathbb{P}^2_{\mathbb{C}}$. Suppose that $\mathcal{G}$ is defined by a projective 1-form ω and let $F\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ be a dominant rational map of degree m. Then, $F^*\omega$ defines a foliation of degree $d = (e+2)m - 2$ on $\mathbb{P}^n_{\mathbb{C}}$. Denote by $\mathrm{PB}(m, e, n)$ the set of foliations of this type. Then, the closure $\overline{\mathrm{PB}(m, e, n)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{(e+2)m-2}(\mathbb{P}^n_{\mathbb{C}})$.[13]

[12] Gómez-Mont, Lins Neto - **Structural stability of foliations with a meromorphic first integral**

[13] Cerveau, Lins Neto and Edixhoven - **Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$**

Components of degree $d \geq 3$

- **Logarithmic components**: Let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r$ homogeneous polynomials with $d_i = \deg(F_i)$. Suppose that $F_1, \ldots, F_r$ are irreducible and coprime. Let $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ such that $\sum_{i=1}^r \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_{r-1} F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

The 1-form Ω defines a $\mathcal{F}_\Omega$ codimension one foliation of degree $d = \sum_i d_i - 2$ on $\mathbb{P}^{\mathrm{n}}_{\mathbb{C}}$. We say that $\mathcal{F}_\Omega$ is a **logarithmic foliation** of type $(d_1, \ldots, d_r)$. Denote by $\mathrm{Log}_n(d_1, \ldots, d_r)$ the set of logarithmic foliations on $\mathbb{P}^{\mathrm{n}}_{\mathbb{C}}$ of type $(d_1, \ldots, d_r)$. Then, the closure $\overline{\mathrm{Log}_n(d_1, \ldots, d_r)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^{\mathrm{n}}_{\mathbb{C}})$.[14][15]

[14] O.Calvo-Andrade - **Irreducible components of the space of foliations**

[15] F. Cukierman, J. Gargiulo and C. D. Massri - **Stability of logarithmic differential one-forms**

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Next: use foliation over positive characteristic to construct new irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

[16] R.C Costa, R. Lizarbe and J.V Pereira - **Codimension one foliations of degree three on projective spaces**

New irreducible components of $\mathbb{F}\mathrm{ol}_d(\mathbb{P}^3_{\mathbb{C}})$

Next: use foliation over positive characteristic to construct new irreducible components of $\mathbb{F}\mathrm{ol}_d(\mathbb{P}^3_{\mathbb{C}})$.

$\mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$ = the collection of rational maps of $\mathbb{P}^3_{\mathbb{C}}$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of degree one.

[16] R.C Costa, R. Lizarbe and J.V Pereira - **Codimension one foliations of degree three on projective spaces**

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Next: use foliation over positive characteristic to construct new irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

$\mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$ = the collection of rational maps of $\mathbb{P}^3_{\mathbb{C}}$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of degree one.

Given $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ let $d = d_1 + d_2 + 2$ and consider the rational map

$$\Psi_{(d;d_1,d_2)} : \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$

$$(\Phi, \mathcal{G}) \longmapsto \Phi^*\mathcal{G}$$

[16] R.C Costa, R. Lizarbe and J.V Pereira - **Codimension one foliations of degree three on projective spaces**

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Next: use foliation over positive characteristic to construct new irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

$\mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$ = the collection of rational maps of $\mathbb{P}^3_{\mathbb{C}}$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of degree one.

Given $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ let $d = d_1 + d_2 + 2$ and consider the rational map

$$\Psi_{(d;d_1,d_2)} : \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$

$$(\Phi, \mathcal{G}) \longmapsto \Phi^*\mathcal{G}$$

Theorem A

Let $C_{(d;d_1,d_2)}$ the image $\Psi_{(d;d_1,d_2)}$. Then $C_{(d;d_1,d_2)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

[16] R.C Costa, R. Lizarbe and J.V Pereira - **Codimension one foliations of degree three on projective spaces**

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Next: use foliation over positive characteristic to construct new irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

$\mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) =$ the collection of rational maps of $\mathbb{P}^3_{\mathbb{C}}$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of degree one.

Given $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ let $d = d_1 + d_2 + 2$ and consider the rational map

$$\Psi_{(d;d_1,d_2)} : \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$

$$(\Phi, \mathcal{G}) \longmapsto \Phi^*\mathcal{G}$$

Theorem A

Let $C_{(d;d_1,d_2)}$ the image $\Psi_{(d;d_1,d_2)}$. Then $C_{(d;d_1,d_2)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

This result generalizes a component of degree $d = 3$ found by R.C Costa, R. Lizarbe e J.V Pereira.[16]

[16] R.C Costa, R. Lizarbe and J.V Pereira - **Codimension one foliations of degree three on projective spaces**

The p-divisor: behavior of the degree

first step: analyze the behavior of the p-divisor on open sets.

The p-divisor: behavior of the degree

first step: analyze the behavior of the p-divisor on open sets.

Theorem

Let $\mathcal{F}$ be a codimension one foliation of degree d on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- *$\mathcal{F}$ is not p-closed with **reduced** p-divisor.*
- *The p-foliation $\mathcal{C}_{\mathcal{F}}$ has degree $e \in \mathbb{Z}_{\geq 0}$.*

The p-divisor: behavior of the degree

first step: analyze the behavior of the p-divisor on open sets.

Theorem

Let $\mathcal{F}$ be a codimension one foliation of degree d on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- *$\mathcal{F}$ is not p-closed with **reduced** p-divisor.*
- *The p-foliation $\mathcal{C}_{\mathcal{F}}$ has degree $e \in \mathbb{Z}_{\geq 0}$.*

Then, there exists an open set $U_{\mathcal{F}}$ in the space of codimension one foliation of degree d on $\mathbb{P}^3_{\mathrm{k}}$ which contains $\mathcal{F}$ such that for any foliation $\mathcal{F}' \in U_{\mathcal{F}}$ we have

1. *$\mathcal{F}'$ is not p-closed, $\Delta_{\mathcal{F}'}$ has no p-factors and $\deg(\Delta_{\mathcal{F}'}) \geq \deg(\Delta_{\mathcal{F}})$.*
2. *The p-foliation $\mathcal{C}_{\mathcal{F}'}$ has degree $e' \leq e$.*

The p-divisor: behavior of the degree

The proof consists in reducing to a problem about polynomials. It is a consequence of the invariance property of the p-divisor and of the following proposition.

The p-divisor: behavior of the degree

The proof consists in reducing to a problem about polynomials. It is a consequence of the invariance property of the p-divisor and of the following proposition.

Proposition

Let $d \in \mathbb{Z}_{>0}$ and k be a field of characteristic $p > 0$. Consider $\mathbb{P}_{\mathrm{k}}^{M_d}$ the projective space parameterizing homogeneous polynomials of degree d in the variables: x_0, x_1, x_2, x_3. Let $G \in \mathrm{k}[x_0, x_1, x_2, x_3]_d$ such that $G = FE^p$ with F free of p-powers. Then, there exists a open set around $[G]$ such that for all $[\tilde{G}] \in U_G$ we have $\tilde{G} = \tilde{F}\tilde{E}^p$ with $\tilde{F}$ free of p-powers with $\deg(\tilde{F}) \geq \deg(F)$.

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$

- k = field of characteristic $p > d + 2$
- $\mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = space parameterizing the foliations on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ with canonical divisor of type (d_1, d_2)
- $\mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = collection of rational maps of degree 1

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$

- k = field of characteristic $p > d+2$
- $\mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = space parameterizing the foliations on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ with canonical divisor of type (d_1, d_2)
- $\mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = collection of rational maps of degree 1

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, $d = d_1 + d_2 + 2$ and consider the rational map

$$\begin{aligned} \Psi_{(d;d_1,d_2)} : \mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}) &\dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}}) \\ (\Phi, \mathcal{G}) &\longmapsto \Phi^*\mathcal{G}. \end{aligned}$$

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$

- k = field of characteristic $p > d+2$
- $\mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = space parameterizing the foliations on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ with canonical divisor of type (d_1, d_2)
- $\mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ = collection of rational maps of degree 1

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, $d = d_1 + d_2 + 2$ and consider the rational map

$$\begin{aligned}\Psi_{(d;d_1,d_2)} : \mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}) &\dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}}) \\ (\Phi, \mathcal{G}) &\longmapsto \Phi^*\mathcal{G}.\end{aligned}$$

Theorem B

Let $X_{(d;d_1,d_2)}$ the Zariski closure of the image of $\Psi_{(d;d_1,d_2)}$. Then $X_{(d;d_1,d_2)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$.

proof(sketch)

step 1: Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- $\mathcal{F} = \Phi^*\mathcal{G}$ for some foliation $\mathcal{G}$ of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ where Φ is the rational map: $[x_0 : x_1 : y_0 : y_1] \mapsto ([x_0 : x_1], [y_0 : y_1])$,
- $\mathcal{G}$ is not p-closed with $\Delta_{\mathcal{G}}$ reduced.

proof(sketch)

step 1: Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- $\mathcal{F} = \Phi^*\mathcal{G}$ for some foliation $\mathcal{G}$ of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ where Φ is the rational map: $[x_0 : x_1 : y_0 : y_1] \mapsto ([x_0 : x_1], [y_0 : y_1])$,
- $\mathcal{G}$ is not p-closed with $\Delta_{\mathcal{G}}$ reduced.

step 2: Note that $\mathcal{F}$ is not p-closed and we have that $\Delta_{\mathcal{F}} = \Phi^*\Delta_{\mathcal{G}}$ (degree comparison). Let T be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$ that contains the image of $\Psi_{(d;d_1,d_2)}$ and $\{\mathcal{F}_t\}_{t\in T}$ the family parametrized by T with $\mathcal{F}_0 = \mathcal{F}$.

proof(sketch)

step 1: Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- $\mathcal{F} = \Phi^*\mathcal{G}$ for some foliation $\mathcal{G}$ of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ where Φ is the rational map: $[x_0 : x_1 : y_0 : y_1] \mapsto ([x_0 : x_1], [y_0 : y_1])$,
- $\mathcal{G}$ is not p-closed with $\Delta_{\mathcal{G}}$ reduced.

step 2: Note that $\mathcal{F}$ is not p-closed and we have that $\Delta_{\mathcal{F}} = \Phi^*\Delta_{\mathcal{G}}$ (degree comparison). Let T be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$ that contains the image of $\Psi_{(d;d_1,d_2)}$ and $\{\mathcal{F}_t\}_{t\in T}$ the family parametrized by T with $\mathcal{F}_0 = \mathcal{F}$.

step 3: Let U be an open set of T that contains 0 such that for all $t \in U$ the foliation $\mathcal{F}_t$ is not p-closed. In this case, for all $t \in U$ we have the associated p-foliation: $\{\mathcal{C}_{\mathcal{F}_t}\}_{t\in U}$. The behavior of the degree of the p-divisor in neighborhood implies that we can assume $\deg(\mathcal{C}_{\mathcal{F}_t}) \leq 1$ for all $t \in U$.

proof(sketch)

step 1: Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^3_{\mathrm{k}}$ and suppose that

- $\mathcal{F} = \Phi^*\mathcal{G}$ for some foliation $\mathcal{G}$ of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ where Φ is the rational map: $[x_0 : x_1 : y_0 : y_1] \mapsto ([x_0 : x_1], [y_0 : y_1])$,
- $\mathcal{G}$ is not p-closed with $\Delta_{\mathcal{G}}$ reduced.

step 2: Note that $\mathcal{F}$ is not p-closed and we have that $\Delta_{\mathcal{F}} = \Phi^*\Delta_{\mathcal{G}}$ (degree comparison). Let T be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathrm{k}})$ that contains the image of $\Psi_{(d;d_1,d_2)}$ and $\{\mathcal{F}_t\}_{t\in T}$ the family parametrized by T with $\mathcal{F}_0 = \mathcal{F}$.

step 3: Let U be an open set of T that contains 0 such that for all $t \in U$ the foliation $\mathcal{F}_t$ is not p-closed. In this case, for all $t \in U$ we have the associated p-foliation: $\{\mathcal{C}_{\mathcal{F}_t}\}_{t\in U}$. The behavior of the degree of the p-divisor in neighborhood implies that we can assume $\deg(\mathcal{C}_{\mathcal{F}_t}) \leq 1$ for all $t \in U$.

step 4: comparison degree shows that $\mathcal{F}$ is not in the linear pull back component. By reducing the open set U we can assume that $\deg(\mathcal{C}_{\mathcal{F}_t}) = 1$ on U.

proof(sketch)

step 5: Since $\deg(\mathcal{F}_t) > 2$ we can assume that $\mathcal{C}_{\mathcal{F}_t}$ is p-closed. Indeed, if no then there exists a homogeneous vector fied v_t of degree 1 tangent to $\mathcal{F}_t$ such that $v_t \wedge v_t^p$ is not zero and defines $\mathcal{F}_t$ (degree comparison).

proof(sketch)

step 5: Since $\deg(\mathcal{F}_t) > 2$ we can assume that $\mathcal{C}_{\mathcal{F}_t}$ is p-closed. Indeed, if no then there exists a homogeneous vector fied v_t of degree 1 tangent to $\mathcal{F}_t$ such that $v_t \wedge v_t^p$ is not zero and defines $\mathcal{F}_t$ (degree comparison).

step 6: By a technical lemma, we ensure that $\mathcal{F}_t$ is a pullback by a rational map of degree 1 of a foliation of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$.

proof(sketch)

step 5: Since $\deg(\mathcal{F}_t) > 2$ we can assume that $\mathcal{C}_{\mathcal{F}_t}$ is p-closed. Indeed, if no then there exists a homogeneous vector fied v_t of degree 1 tangent to $\mathcal{F}_t$ such that $v_t \wedge v_t^p$ is not zero and defines $\mathcal{F}_t$ (degree comparison).

step 6: By a technical lemma, we ensure that $\mathcal{F}_t$ is a pullback by a rational map of degree 1 of a foliation of type (d_1, d_2) on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$.

step 7: So, there is a open set $U_{\mathcal{F}}$ in the space of codimension one foliation of degree d on $\mathbb{P}^{\mathrm{n}}_{\mathrm{k}}$ that contains $\mathcal{F}$ which has the following property:

- For all foliation $\tilde{\mathcal{F}} \in U_{\mathcal{F}}$ we have $\tilde{\mathcal{F}} = \gamma^* \tilde{\mathcal{G}}$ for some $\tilde{\mathcal{G}} \in \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$ and $\gamma \in \mathrm{Map}_1(\mathbb{P}^3_{\mathrm{k}}, \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}})$.

So, $U_{\mathcal{F}} \subset X_{(d;d_1,d_2)}$ and by considering the Zariski closure we conclude that $T = X_{(d;d_1,d_2)}$.

Redution mod p

- $X = \mathcal{Z}(F_0, \ldots, F_r) \subset \mathbb{P}^M_{\mathbb{C}}$ irreducible variety.
- $R =$ finitely generated $\mathbb{Z}$-algebra obtained by adjunction of all coefficient that occurs in $F_0, \ldots, F_r$.

For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(R)$ of R the field $k(\mathfrak{p}) = R/\mathfrak{p}$ is finite, in particular, of characteristic $p > 0$.

Redution mod p

- $X = \mathcal{Z}(F_0, \ldots, F_r) \subset \mathbb{P}^M_{\mathbb{C}}$ irreducible variety.
- $R =$ finitely generated $\mathbb{Z}$-algebra obtained by adjunction of all coefficient that occurs in $F_0, \ldots, F_r$.

For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(R)$ of R the field $k(\mathfrak{p}) = R/\mathfrak{p}$ is finite, in particular, of characteristic $p > 0$.

Proposition (Bertini-Noether)

Let $\mathfrak{p} \in \boldsymbol{Spm}(R)$ be a maximal ideal of R and consider $X_{\mathfrak{p}}$ the variety defined over $\overline{k(\mathfrak{p})}$ obtained via reduction modulo $\mathfrak{p}$ of $F_0, \ldots, F_r$. Then $X_{\mathfrak{p}}$ is irreducible and $\dim X = \dim X_{\mathfrak{p}}$ for almost all maximal ideals of R, i.e for all maximal ideals of R outside a proper closed subset $E \subset \boldsymbol{Spm}(R)$.

Irreducible components and reduction mod p

Let X be a projective variety on $\mathbb{P}^M_{\mathbb{C}}$ given by polynomials $F_0, \ldots, F_r \in \mathbb{C}[x_0, \ldots, x_M]$ and $Y \subset X$ be an irreducible closed given by $H_0, \ldots, H_k \in \mathbb{C}[x_0, \ldots, x_M]$. Let Z be an irreducible component of X which contains Y and suppose that it given by polynomials $G_0, \ldots, G_l$. Denote by R the a finitely generated $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appers in

$$F_0, \ldots, F_r, G_0, \ldots, G_l, H_0, \ldots, H_k.$$

Irreducible components and reduction mod p

Let X be a projective variety on $\mathbb{P}^M_{\mathbb{C}}$ given by polynomials $F_0, \ldots, F_r \in \mathbb{C}[x_0, \ldots, x_M]$ and $Y \subset X$ be an irreducible closed given by $H_0, \ldots, H_k \in \mathbb{C}[x_0, \ldots, x_M]$. Let Z be an irreducible component of X which contains Y and suppose that it given by polynomials $G_0, \ldots, G_l$. Denote by R the a finitely generated $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appers in

$$F_0, \ldots, F_r, G_0, \ldots, G_l, H_0, \ldots, H_k.$$

Corollary

Suppose that there is a dense set S of $\mathbf{Spm}(R)$ such that $Y_{\mathfrak{p}} = Z_{\mathfrak{p}}$ for all ideal $\mathfrak{p} \in S$. Then $Y = Z$ and so Y is an irreducible component of X.

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Consider the rational map

$$\Psi_{(d;d_1,d_2)}\colon \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$
$$(\Phi, \mathcal{G}) \mapsto \Phi^*\mathcal{G}.$$

New irreducible components of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$

Consider the rational map

$$\Psi_{(d;d_1,d_2)}\colon \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$
$$(\Phi, \mathcal{G}) \mapsto \Phi^*\mathcal{G}.$$

Theorem A

Let $C_{(d;d_1,d_2)}$ be the Zariski closure of the image $\Psi_{(d;d_1,d_2)}$. Then $C_{(d;d_1,d_2)}$ is an irreducible component $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

Recall: the result over characteristic $p > d + 2$ is true.

Proof of Theorem A

- **step I:** Let Z be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$ which contains $C_{(d;d_1,d_2)}$ and let $\{E_0,\ldots,E_h\}$ be the union of a collection of polynomials which describes the varieties: $Z, C_{(d;d_1,d_2)}$ and $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

[17] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

Proof of Theorem A

- **step I:** Let Z be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$ which contains $C_{(d;d_1,d_2)}$ and let $\{E_0,\ldots,E_h\}$ be the union of a collection of polynomials which describes the varieties: $Z, C_{(d;d_1,d_2)}$ and $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.
- **step II:** Let R be a $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appears in $E_0,\ldots,E_h$ and T the closed set in $\mathbf{Spm}(R)$ given by $\cup_{j=2}^{d+2} V(jR) \subset \mathbf{Spm}(R)$. The Theorem B (component in positive characteristic) ensures that for all $\mathfrak{p} \in \mathbf{Spm}(R) - T$ we have $C_{(d;d_1,d_2),\mathfrak{p}} = Z_{\mathfrak{p}}$.

[17] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

Proof of Theorem A

- **step I:** Let Z be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$ which contains $C_{(d;d_1,d_2)}$ and let $\{E_0,\ldots,E_h\}$ be the union of a collection of polynomials which describes the varieties: $Z, C_{(d;d_1,d_2)}$ and $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.
- **step II:** Let R be a $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appears in $E_0,\ldots,E_h$ and T the closed set in $\mathbf{Spm}(R)$ given by $\cup_{j=2}^{d+2} V(jR) \subset \mathbf{Spm}(R)$. The Theorem B (component in positive characteristic) ensures that for all $\mathfrak{p} \in \mathbf{Spm}(R) - T$ we have $C_{(d;d_1,d_2),\mathfrak{p}} = Z_{\mathfrak{p}}$.
- **step III:** By the precedent corollary we conclude that $Z = C_{(d;d_1,d_2)}$ is a irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

[17] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

Proof of Theorem A

- **step I:** Let Z be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$ which contains $C_{(d;d_1,d_2)}$ and let $\{E_0, \dots, E_h\}$ be the union of a collection of polynomials which describes the varieties: $Z, C_{(d;d_1,d_2)}$ and $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.
- **step II:** Let R be a $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appears in $E_0, \dots, E_h$ and T the closed set in $\mathbf{Spm}(R)$ given by $\cup_{j=2}^{d+2} V(jR) \subset \mathbf{Spm}(R)$. The Theorem B (component in positive characteristic) ensures that for all $\mathfrak{p} \in \mathbf{Spm}(R) - T$ we have $C_{(d;d_1,d_2),\mathfrak{p}} = Z_{\mathfrak{p}}$.
- **step III:** By the precedent corollary we conclude that $Z = C_{(d;d_1,d_2)}$ is a irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

By using similar ideas the result generalizes to $\mathbb{P}^n_{\mathbb{C}}$ for $n \geq 3$[17]

[17] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

Proof of Theorem A

- **step I:** Let Z be an irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$ which contains $C_{(d;d_1,d_2)}$ and let $\{E_0,\ldots,E_h\}$ be the union of a collection of polynomials which describes the varieties: $Z, C_{(d;d_1,d_2)}$ and $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.
- **step II:** Let R be a $\mathbb{Z}$-algebra obtained by adjunction of all coefficients which appears in $E_0,\ldots,E_h$ and T the closed set in $\mathbf{Spm}(R)$ given by $\cup_{j=2}^{d+2} V(jR) \subset \mathbf{Spm}(R)$. The Theorem B (component in positive characteristic) ensures that for all $\mathfrak{p} \in \mathbf{Spm}(R) - T$ we have $C_{(d;d_1,d_2),\mathfrak{p}} = Z_{\mathfrak{p}}$.
- **step III:** By the precedent corollary we conclude that $Z = C_{(d;d_1,d_2)}$ is a irreducible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

By using similar ideas the result generalizes to $\mathbb{P}^n_{\mathbb{C}}$ for $n \geq 3$[17]

Corollary

The space of codimension one holomorphic foliations and degree $d \geq 3$ on $\mathbb{P}^3_{\mathbb{C}}$ has at least $\left\lfloor \frac{d-1}{2} \right\rfloor$ distinct irreducible components whose generic element does not have a polynomial integrating factor.

[17] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

Thank you :-)