On arithmetic of planar vector fields

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Structure

- Part I: Introduction
- Part II: Foliations in characteristic $p > 0$
- Part III: The p-divisor and non-algebraicity
- Part IV: From characteristic 2 to $\mathbb{C}$

Part I: Introduction

Foliations

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A **foliation**, $\mathcal{F}$, of degree d on the projective plane $\mathbb{P}^2_K$ is given, mod K^*, by a non-zero element $\omega \in \mathrm{H}^0(\mathbb{P}^2_K, \Omega^1_{\mathbb{P}^2_K}(d+2))$ with finite singular locus

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Explicitly:

- Using the Euler exact sequence we can see ω as a projective 1-form:

$$\omega = Adx + Bdy + Cdz$$

on $\mathbb{A}^3_K$ such that $A, B, C \in K[x, y, z]$ are homogeneous of degree $d+1$ and $Ax + By + Cz = 0$ with

$$\mathrm{sing}(\omega) = \mathcal{Z}(A, B, C) = \{p \in \mathbb{P}^2_K \mid A(p) = B(p) = C(p) = 0\}$$

finite.

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- In this sense, a foliation of degree d on $\mathbb{P}^2_K$ is determined, modulo K^*, by a homogeneous vector field on $\mathbb{A}^3_K$:

$$v = A_0\partial_x + A_1\partial_y + A_2\partial_z \in \mathfrak{X}_d(\mathbb{A}^3_K)$$

where $A_0, A_1, A_2 \in K[x,y,z]$ are homogeneous of degree d with

$$\mathbf{div}(v) = \partial_x A_0 + \partial_y A_1 + \partial_z A_2 = 0$$

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The equivalence of these notions is given by the following result:

Proposition

[a] *There is a bijection between the set of projective 1-forms on $\mathbb{A}^3_K$ of degree $d+1$ and homogeneous vector fields with divergent zero of degree d.*

[a] Jouanolou - **Equations de Pfaff algébriques**

Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

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and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

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Example: Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then ω defines a foliation of degree 1 on $\mathbb{P}^2_K$ and the vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2-\alpha}{3}\right) y\partial_y + \left(\frac{-1-\alpha}{3}\right) z\partial_z$$

Invariant curves

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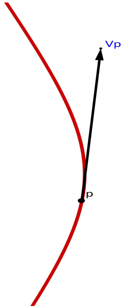

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- **Logarithmic foliations**: Let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r \in K[x, y, z]$ homogeneous polynomials with $d_i = \deg(F_i)$. Suppose that $F_1, \ldots, F_r$ are irreducible and coprime. Let $\alpha_1, \ldots, \alpha_r \in K^*$ such that $\sum_{i=1}^{r} \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_{r-1} F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

  The 1-form Ω defines, $\mathcal{F}_\Omega$, a foliation of degree $d = \sum_i d_i - 2$ on $\mathbb{P}^2_K$. We say that $\mathcal{F}_\Omega$ is a **logarithmic foliation** of type $(d_1, \ldots, d_r)$. The curves $C_i = \{F_i = 0\}$ are $\mathcal{F}_\Omega$-invariant.

Jouanolou example: foliations without invariant curves

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the projective 1-form:

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

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Theorem (Jouanolou)

[a] *If $K = \mathbb{C}$ the foliation $\mathcal{F}_d$ does not have invariant algebraic curves.*

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This result implies in particular that on $\mathbb{P}^2_{\mathbb{C}}$ **almost all** foliation on the complex projective plane have no algebraic invariant curves.

Part II: Foliations in characteristic $p > 0$

Foliations on characteristic $p > 0$: the p-divisor on $\mathbb{P}^2_K$

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The p-**divisor** is given by

$$\Delta_{\mathcal{F}} = \{i_{v_\omega^p}\omega = 0\}.$$

Note that $\Delta_{\mathcal{F}}$ has degree $p(d-1)+d+2$.

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Definition

*The foliation $\mathcal{F}$ is **p-closed** if $\Delta_{\mathcal{F}} = 0$.*

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By iteration we get:

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha^p}{3}\right) y\partial_y + \left(\frac{-1 - \alpha^p}{3}\right) z\partial_z$$

and the equation for the p-divisor is:

$$i_{v^p}\omega = yzv^p(x) - \alpha xzv^p(y) + (\alpha - 1)xyv^p(z) = (\alpha^p - \alpha)xyz$$

If $\alpha \notin \mathbb{F}_p$:

$$\Delta_{\mathcal{F}} = \{x = 0\} + \{y = 0\} + \{z = 0\}.$$

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Proposition

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ be an algebraic curve.*

- *If C is $\mathcal{F}$-invariant then* $\mathrm{ord}_C(\Delta_{\mathcal{F}}) > 0$;

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[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

Corollary

On the projective plane over characteristic $p > 0$ any foliation of degree d such that $p \nmid d + 2$ has an invariant algebraic curve.

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On the projective plane over characteristic $p > 0$ any non-p-closed foliation of degree d has an invariant algebraic curve of degree less than or equal to $p(d-1)+d+2$.

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Proposition

[a] *A foliation is p-closed if and only if it has infinitely many solutions.*

[a] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

Jouanolou foliation in positive characteristic

Theorem

Let $p > 2$ be a prime number such that $7 \nmid p + 4$ and such that $p \not\equiv 1 \mod 3$. Then, the Jouanolou foliation of degree two $\mathcal{F}_2$ defined over a field of characteristic p has irreducible p-divisor.

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Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ defined by a projective 1-form ω. An automorphism of a foliation $\mathcal{F}$ is a automorphism Φ of $\mathbb{P}^2_K$ which preserves $\mathcal{F}$:

$$\Phi^*\omega = \sigma(\Phi)\omega$$

for some $\sigma(\Phi) \in K^*$. The automorphism group of $\mathcal{F}$ is denoted by $\mathrm{Aut}(\mathcal{F})$.

Jouanolou foliation: example

Let $\mathcal{F}_d$ defined by

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Let γ be a primitive $(d^2 + d + 1)$th root[1] of unity and consider

$$\Phi \colon \mathbb{P}^2_K \longrightarrow \mathbb{P}^2_K \qquad [x : y : z] \mapsto [\gamma^{d^2+1} x : \gamma y : z]$$

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Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ non-p-closed and $\Phi \in \mathrm{Aut}(\mathcal{F})$. Then $\Phi^* \Delta_{\mathcal{F}} = \Delta_{\mathcal{F}}$.

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Case $d = 2$

$$\mathcal{F}_2 \colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the p-divisor of $\mathcal{J} := \mathcal{F}_2$:

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  1. some power Φ^l fixes the curve C: computation implies that $\beta = 0$, contradiction;
  2. Φ^l does not fix C for all $l \in \{1, \dots, 6\}$: In that case for any prime divisor P in the support of $\Delta_{\mathcal{J}}$ we have: $P, \Phi^* P, \dots, (\Phi^6)^* P$ are disticts and this implies that $7 \mid p+4$, a contradiction;

Jouanolou foliation II

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- $p < d$ *and* $p \not\equiv 1 \mod 3$;
- $d^2 + d + 1$ *is prime.*

Then the Jouanolou foliation $\mathcal{F}_d$ has irreducible p-divisor ***or*** $\Delta_{\mathcal{F}_d} = C + pR$ *with* $\deg(C) = pl + d + 2$ *with* $l > 0$ *and* R *not* $\mathcal{F}_d$*-invariant.*

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An interesting consequence: the Jouanolou foliation $\mathcal{F}_d$ has a unique algebraic curve.

characteristic $p = 5$ and $d \leq 100$

d	d^2+d+1	$\deg(C)$	l	R
6	43	18	2	$\{xyz = 0\}$
12	157	39	5	$\{xyz = 0\}$
17	307	54	7	$\{xyz = 0\}$
21	463	63	8	$\{xyz = 0\}$
27	757	84	11	$\{xyz = 0\}$
41	1723	123	16	$\{xyz = 0\}$
57	3307	174	23	$\{xyz = 0\}$
62	3907	189	25	$\{xyz = 0\}$
66	4423	198	26	$\{xyz = 0\}$
71	5113	213	28	$\{xyz = 0\}$
77	6007	234	31	$\{xyz = 0\}$

- for $d \in \{2, 14, 24, 54, 59, 69, 89, 99\}$, $\Delta_{\mathcal{F}_d}$ is irreducible;
- for the remaining cases, $\Delta_{\mathcal{F}_d} = 0$.

Part III: The p-divisor and non-algebraicity

Reduction modulo p

Consider the case where $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree d defined by the projective 1-form:

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Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ given by the 1-form:

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For the Jouanolou foliation, $A, B, C \in \mathbb{Z}[x, y, z]$ so that $\mathbb{Z}[\mathcal{F}_d] = \mathbb{Z}$.

Reduction modulo p

Fact: For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$ the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

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Denote by $\omega_{\mathfrak{p}}$ the 1-form over $\overline{\mathbb{F}}_{\mathfrak{p}}$ obtained by reduction modulo $\mathfrak{p}$ of all coefficient which appears in A, B and C. We obtain a non-zero element of $\mathrm{H}^0(\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \Omega^1_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}}(d+2))$ and $\omega_{\mathfrak{p}}$ determines a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}$:

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Definition

*The foliation determined by $\omega_{\mathfrak{p}}$ is denoted by $\mathcal{F}_{\mathfrak{p}}$ and is called the **reduction modulo** p of $\mathcal{F}$.*

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Problem

Suppose that an abstract property $\boldsymbol{P}$ holds for $\mathcal{F}_{\mathfrak{p}}$ for an infinitely many primes (or almost all primes) $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$. What we can say about $\mathcal{F}$?

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- **infinitely many primes** = primes in some dense subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$;
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When $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}$ then the notions: **infinitely many primes** and **all most primes** are the usual notions.

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The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;

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Proposition

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ and suppose that $\mathcal{F}_{\mathfrak{p}}$ has an invariant curve of degree less than $\boldsymbol{d}$ for almost all primes $\mathfrak{p}$. Then, $\mathcal{F}$ has an invariant curve of degree less than $\boldsymbol{d}$.

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Idea: the set $S(\mathcal{F}, K, d)$ of foliations on $\mathbb{P}^2_K$ that have invariant curves of degree $\leq d$ is algebraic variety over K. In particular, $S(\mathcal{F}, \mathbb{C}, d) \neq \varnothing$ if and only if $S(\mathcal{F}, \overline{\mathbb{F}}_{\mathfrak{p}}, d) \neq \varnothing$ for almost all primes $\mathfrak{p}$.

Algebraic solutions

Goal: use reduction modulo p to prove the non-algebraicity of holomorphic foliations

[2] Carnicer - **The Poincare problem in the nondicritical case**

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Proposition

[a] *Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form $\omega = Adx + Bdy + Cdz$ with $A, B, C \in \mathbb{Z}[x, y, z]$. Let p be a prime number such that $p > d + 2$. If $\Delta_{\mathcal{F}_p}$ is irreducible then $\mathcal{F}$ has no algebraic solutions.*

[a] W.Mendson - **Foliations on smooth algebraic surfaces in position characteristic**

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[a] W.Mendson - **Foliations on smooth algebraic surfaces in position characteristic**

Idea: Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. We can assume that $F \in \mathbb{Z}[x, y, z]$. The Carnicer bound[2] implies that $\deg(C) \leq d + 2$. Reducing modulo p and using the irreducibility of $\Delta_{\mathcal{F}_p}$ we get a contradiction.

[2] Carnicer - **The Poincare problem in the nondicritical case**

Applications

Corollary

The Jouanolou foliation of degree 2 or 3 has no algebraic solutions.

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The Jouanolou foliation of degree 2 or 3 has no algebraic solutions.

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Idea: Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. This curve has degree e. For large primes p we obtain $C \mod p = \Delta_{\mathcal{F}_p}$, a contradiction since the degree of the p-divisor depends of p.

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Let Φ be the morphism

$$\Phi\colon \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad [x : y : z] \mapsto [x^2 : y^2 : z^2]$$

and consider $\mathcal{G} = \Phi^*\mathcal{F}_d$, where $\mathcal{F}_d$ is the Jouanolou foliation of degree d. Then, $\mathcal{G}_{\mathfrak{p}}$ is not p-closed for infinitely many primes $\mathfrak{p}$ with the p-divisor having a p-component.

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and consider $\mathcal{G} = \Phi^* \mathcal{F}_d$*, where* $\mathcal{F}_d$ *is the Jouanolou foliation of degree* d*. Then,* $\mathcal{G}_{\mathfrak{p}}$ *is not* p*-closed for infinitely many primes* $\mathfrak{p}$ *with the* p*-divisor having a* p*-component.*

So, there are foliations on $\mathbb{P}^2_{\mathbb{C}}$ without algebraic invariant curves such that the reduction modulo p has non-irreducible p-divisor for infinitely many primes p.

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So, there are foliations on $\mathbb{P}^2_{\mathbb{C}}$ without algebraic invariant curves such that the reduction modulo p has non-irreducible p-divisor for infinitely many primes p. Related to this problem we have the following:

Problem

[a] *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ and $C = \{F = 0\}$ be an irreducible $\mathcal{F}$-invariant curve. When $\mathrm{sing}(C) \subset \mathrm{sing}(\mathcal{F})$? Does this occur if $\mathcal{F}$ is p-reduced with $\deg(C) < p$?*

[a] Recently, F.Touzet showed me an argument for the case: $\deg(C)(\deg(C) - 1) < p$

Examples

Example

Assume that K *has characteristic* 3 *and consider the Jouanolou foliation of degree* 2 *over* K. *Then, the* p*-divisor* Δ *is irreducible with* $\operatorname{sing}(\Delta) \not\subset \operatorname{sing}(\mathcal{F})$.

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Example (F.Touzet)

Consider the foliation 5*-closed foliation given by:*

$$\omega = 2z(x+y)dx + z(2z+x)dy + 4(2x^2 + 3xy + 2yz)dz$$

The curve $C = \{-x^4 + x^3y + x^2yz + y^2z^2\}$ *is* $\mathcal{F}$*-invariant with* $[0:0:1] \in \mathrm{sing}(C)$ *but not in* $\mathrm{sing}(\mathcal{F})$*.*

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Consider the question: assume that $\mathcal{F}$ is p-reduced and $\deg(C) < p$. Is it true that $\operatorname{sing}(C) \subset \operatorname{sing}(\mathcal{F})$?

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Proposition

Assume that the answer is YES. Let $\mathcal{F}$ be a good[a] foliation of degree two. Then $\mathcal{F}$ has not algebraic solutions if and only if $\Delta_{\mathcal{F}_p}$ is irreducible for infinitely many primes p.

[a] $= \mathcal{F} \mod p$ is p-reduced for infinitely many primes p

Part IV: From characteristic 2 to $\mathbb{C}$

Jouanolou foliation in characteristic 2

Proposition

$\mathcal{F}_d$ is 2-closed if and only $d \equiv 0 \mod 2$. If $d \equiv 1 \mod 2$ then $\Delta_{\mathcal{F}_d}$ is irreducible of degree $3d$.

[3] S.Gao - **Absolute irreducibility of polynomials via Newton polytopes**

[4] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**

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By explicit computation, we can check that the 2-divisor is irreducible[3] for $d \equiv 1 \mod 2$:

$$\Delta_{\mathcal{F}_d} = y^{2d+1}z^{d-1} + x^d y^d z^d + x^{2d+1}y^{d-1} + x^{d-1}z^{2d+1}$$

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Using reduction modulo two it is possible to give a new proof of the following result[4]:

Theorem

Let $d \in \mathbb{Z}$ such that $d \not\equiv 1 \mod 3$ and $d \equiv 1 \mod 2$. If $K = \mathbb{C}$ then the Jouanolou foliation of degree d has no algebraic solutions.

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Invariant curves via characteristic 2

The idea of the proof:

[5] J.V.Pereira, P.F.Sánchez - **Automorphism and non-integrability**

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step 1: Suppose that there is an algebraic curve C_0 given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$;

[5] J.V.Pereira, P.F.Sánchez - **Automorphism and non-integrability**

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The idea of the proof:

step 1: Suppose that there is an algebraic curve C_0 given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$;

step 2: An particular property of the automorphism group of $\mathcal{F}_d$[5] permits to construct a invariant curve C_1, from C_0, that has degree $d + 2$ and it is defined over a number field L. In particular, $\deg(C_1) \equiv 1 \mod 2$;

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step 3: The condition $d \equiv 1 \mod 2$ implies that the foliation $\mathcal{F}_d \mod 2\mathbb{Z}$ is not 2-closed. The 2-divisor is irreducible of degree $3d$;

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step 4: Reducing C_1 modulo 2 we get a contradiction by degree comparison since $C_1 = \Delta_{\mathcal{F}_d} \mod 2$.

[5] J.V.Pereira, P.F.Sánchez - **Automorphism and non-integrability**

Thank you ;-)