# The Jouanolou foliation in positive characteristic

Wodson Mendson

**Universidade Federal Fluminense - UFF**
**LEGAL**

March 15, 2024

# Structure

- Part I: Introduction
- Part II: Foliations in characteristic $p > 0$;
- Part III: The Jouanolou foliation in positive characteristic

# Part I: Introduction

# Foliations

**In this talk:** foliations = foliations on the projective plane

$K$ = algebraically closed field

# Foliations

**In this talk:** foliations = foliations on the projective plane

$K$ = algebraically closed field

Let $d \in \mathbb{Z}_{>0}$

A **foliation**, $\mathcal{F}$, of degree $d$ on the projective plane $\mathbb{P}^2_K$ is given, mod $K^*$, by a non-zero element $\omega \in \mathrm{H}^0(\mathbb{P}^2_K, \Omega^1_{\mathbb{P}^2_K}(d+2))$ with finite singular locus

# Foliations

**In this talk:** foliations = foliations on the projective plane

$K$ = algebraically closed field

Let $d \in \mathbb{Z}_{>0}$

A **foliation**, $\mathcal{F}$, of degree $d$ on the projective plane $\mathbb{P}^2_K$ is given, mod $K^*$, by a non-zero element $\omega \in \mathrm{H}^0(\mathbb{P}^2_K, \Omega^1_{\mathbb{P}^2_K}(d+2))$ with finite singular locus

**Explicitly:**

- Using the Euler exact sequence we can see $\omega$ as a projective 1-form:

$$\omega = Adx + Bdy + Cdz$$

on $\mathbb{A}^3_K$ such that $A, B, C \in K[x, y, z]$ are homogeneous of degree $d + 1$ and $Ax + By + Cz = 0$ with

$$\mathrm{sing}(\omega) = \mathcal{Z}(A, B, C) = \{p \in \mathbb{P}^2_K \mid A(p) = B(p) = C(p) = 0\}$$

finite.

# Foliation in terms of vector field

Assume that the characteristic of $K$ does not divide $d + 2$. The notion of foliation can be given in terms of homogeneous vector fields.

# Foliation in terms of vector field

Assume that the characteristic of $K$ does not divide $d+2$. The notion of foliation can be given in terms of homogeneous vector fields.

- In this sense, a foliation of degree $d$ on $\mathbb{P}^2_K$ is determined, modulo $K^*$, by a homogeneous vector field on $\mathbb{A}^3_K$:

$$v = A_0\partial_x + A_1\partial_y + A_2\partial_z \in \mathfrak{X}_d(\mathbb{A}^3_K)$$

where $A_0, A_1, A_2 \in K[x, y, z]$ are homogeneous of degree $d$ with

$$\mathbf{div}(v) = \partial_x A_0 + \partial_y A_1 + \partial_z A_2 = 0$$

# Foliation in terms of vector field

Assume that the characteristic of $K$ does not divide $d+2$. The notion of foliation can be given in terms of homogeneous vector fields.

- In this sense, a foliation of degree $d$ on $\mathbb{P}^2_K$ is determined, modulo $K^*$, by a homogeneous vector field on $\mathbb{A}^3_K$:

$$v = A_0\partial_x + A_1\partial_y + A_2\partial_z \in \mathfrak{X}_d(\mathbb{A}^3_K)$$

where $A_0, A_1, A_2 \in K[x, y, z]$ are homogeneous of degree $d$ with

$$\mathbf{div}(v) = \partial_x A_0 + \partial_y A_1 + \partial_z A_2 = 0$$

The equivalence of these notions is given by the following result:

**Proposition**

[a] *There is a bijection between the set of projective 1-forms on $\mathbb{A}^3_K$ of degree $d+1$ and homogeneous vector fields with divergent zero of degree $d$.*

[a] Jouanolou - **Equations de Pfaff algébriques**

# Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

$$\omega = Adx + Bdy + Cdz$$

and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

# Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

$$\omega = Adx + Bdy + Cdz$$

and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

The homogeneous vector field degree $d$ with divergent zero is given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

# Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

$$\omega = Adx + Bdy + Cdz$$

and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

The homogeneous vector field degree $d$ with divergent zero is given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

**Example:** Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

# Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

$$\omega = Adx + Bdy + Cdz$$

and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

The homogeneous vector field degree $d$ with divergent zero is given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

**Example:** Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then $\omega$ defines a foliation of degree 1 on $\mathbb{P}^2_K$ and the vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2-\alpha}{3}\right) y\partial_y + \left(\frac{-1-\alpha}{3}\right) z\partial_z$$

# Invariant curves

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ given by a projective 1-form $\omega$.

# Invariant curves

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ given by a projective 1-form $\omega$.

Let $C = \{F = 0\} \subset \mathbb{P}^2_K$ be an algebraic curve given by a irreducible polynomial $F \in K[x, y, z]$.

# Invariant curves

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ given by a projective 1-form $\omega$.

Let $C = \{F = 0\} \subset \mathbb{P}^2_K$ be an algebraic curve given by a irreducible polynomial $F \in K[x, y, z]$.

**Definition**

*The curve $C$ is **$\mathcal{F}$-invariant** if there is a homogeneous 2-form $\sigma$ on $\mathbb{A}^3_K$ such that*

$$dF \wedge \omega = F\sigma$$

# Invariant curves

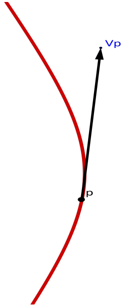
Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ given by a projective 1-form $\omega$.

Let $C = \{F = 0\} \subset \mathbb{P}^2_K$ be an algebraic curve given by a irreducible polynomial $F \in K[x, y, z]$.

**Definition**

*The curve $C$ is **$\mathcal{F}$-invariant** if there is a homogeneous 2-form $\sigma$ on $\mathbb{A}^3_K$ such that*

$$dF \wedge \omega = F\sigma$$



# Example: foliations with invariant curves

- The foliation given by

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

has $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ as invariant curves.

# Example: foliations with invariant curves

- The foliation given by

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

has $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ as invariant curves.

- **Logarithmic foliations**: Let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r \in K[x, y, z]$ homogeneous polynomials with $d_i = \deg(F_i)$. Suppose that $F_1, \ldots, F_r$ are irreducible and coprime. Let $\alpha_1, \ldots, \alpha_r \in K^*$ such that $\sum_{i=1}^r \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_{r-1} F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

The 1-form $\Omega$ defines, $\mathcal{F}_\Omega$, a foliation of degree $d = \sum_i d_i - 2$ on $\mathbb{P}^2_K$. We say that $\mathcal{F}_\Omega$ is a **logarithmic foliation** of type $(d_1, \ldots, d_r)$. The curves $C_i = \{F_i = 0\}$ are $\mathcal{F}_\Omega$-invariant.

# Jouanolou example: foliations without invariant curves

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the projective 1-form:

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$
$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

# Jouanolou example: foliations without invariant curves

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the projective 1-form:

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$
$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

**Theorem (Jouanolou)**

[a] *If $K = \mathbb{C}$ the foliation $\mathcal{F}_d$ does not have invariant algebraic curves.*

[a] Jouanolou - **Equations de Pfaff algébriques**

# Jouanolou example: foliations without invariant curves

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the projective 1-form:

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

**Theorem (Jouanolou)**

[a] *If $K = \mathbb{C}$ the foliation $\mathcal{F}_d$ does not have invariant algebraic curves.*

[a]Jouanolou - **Equations de Pfaff algébriques**

This result implies in particular that on $\mathbb{P}^2_{\mathbb{C}}$ **almost all** foliation on the complex projective plane have no algebraic invariant curves.

# Part II: Foliations in characteristic $p > 0$

# The $p$-divisor on $\mathbb{P}^2_K$

$K =$ algebraically closed field of characteristic $p > 0$.

# The $p$-divisor on $\mathbb{P}^2_K$

$K =$ algebraically closed field of characteristic $p > 0$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ of degree $d$ defined by

$$\omega = Adx + Bdy + Cdz$$

and suppose that $p \nmid d + 2$.

# The $p$-divisor on $\mathbb{P}^2_K$

$K =$ algebraically closed field of characteristic $p > 0$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ of degree $d$ defined by

$$\omega = Adx + Bdy + Cdz$$

and suppose that $p \nmid d + 2$.Write $d\omega = (d + 2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy)$ and let $v_\omega$ be the vector field of degree $d$ associated to $\mathcal{F}$ given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

# The $p$-divisor on $\mathbb{P}^2_K$

$K =$ algebraically closed field of characteristic $p > 0$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ of degree $d$ defined by

$$\omega = Adx + Bdy + Cdz$$

and suppose that $p \nmid d + 2$.Write $d\omega = (d + 2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy)$ and let $v_\omega$ be the vector field of degree $d$ associated to $\mathcal{F}$ given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

The $p$**-divisor** is given by

$$\Delta_{\mathcal{F}} = \{i_{v_\omega^p}\omega = 0\}.$$

Note that $\Delta_{\mathcal{F}}$ has degree $p(d-1) + d + 2$.

# The $p$-divisor on $\mathbb{P}^2_K$

$K =$ algebraically closed field of characteristic $p > 0$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ of degree $d$ defined by

$$\omega = Adx + Bdy + Cdz$$

and suppose that $p \nmid d+2$.Write $d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy)$ and let $v_\omega$ be the vector field of degree $d$ associated to $\mathcal{F}$ given by

$$v_\omega = L\partial_x + M\partial_y + N\partial_z.$$

The ***p*-divisor** is given by

$$\Delta_{\mathcal{F}} = \{i_{v_\omega^p}\omega = 0\}.$$

Note that $\Delta_{\mathcal{F}}$ has degree $p(d-1)+d+2$.

**Definition**

*The foliation $\mathcal{F}$ is **$p$-closed** if $\Delta_{\mathcal{F}} = 0$.*

# Example

Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

# Example

Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then $\omega$ defines a foliation of degree 1 on $\mathbb{P}^2_K$. The vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha}{3}\right) y\partial_y + \left(\frac{-1 - \alpha}{3}\right) z\partial_z$$

# Example

Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then $\omega$ defines a foliation of degree 1 on $\mathbb{P}^2_K$. The vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha}{3}\right) y\partial_y + \left(\frac{-1 - \alpha}{3}\right) z\partial_z$$

By iteration we get:

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha^p}{3}\right) y\partial_y + \left(\frac{-1 - \alpha^p}{3}\right) z\partial_z$$

# Example

Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then $\omega$ defines a foliation of degree 1 on $\mathbb{P}^2_K$. The vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha}{3}\right) y\partial_y + \left(\frac{-1 - \alpha}{3}\right) z\partial_z$$

By iteration we get:

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha^p}{3}\right) y\partial_y + \left(\frac{-1 - \alpha^p}{3}\right) z\partial_z$$

and the equation for the $p$-divisor is:

$$i_{v^p}\omega = yzv^p(x) - \alpha xzv^p(y) + (\alpha - 1)xyv^p(z) = (\alpha^p - \alpha)xyz$$

# Example

Let $\alpha \in K^*$ and consider:

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

Then $\omega$ defines a foliation of degree 1 on $\mathbb{P}^2_K$. The vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha}{3}\right) y\partial_y + \left(\frac{-1 - \alpha}{3}\right) z\partial_z$$

By iteration we get:

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha^p}{3}\right) y\partial_y + \left(\frac{-1 - \alpha^p}{3}\right) z\partial_z$$

and the equation for the $p$-divisor is:

$$i_{v^p}\omega = yzv^p(x) - \alpha xzv^p(y) + (\alpha - 1)xyv^p(z) = (\alpha^p - \alpha)xyz$$

If $\alpha \notin \mathbb{F}_p$ :

$$\Delta_{\mathcal{F}} = \{x = 0\} + \{y = 0\} + \{z = 0\}.$$

## The $p$-divisor

The main property of the $p$-divisor is the following.

## The $p$-divisor

The main property of the $p$-divisor is the following.

**Proposition**

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ be an algebraic curve.*

- *If $C$ is $\mathcal{F}$-invariant then* $\mathrm{ord}_C(\Delta_{\mathcal{F}}) > 0$*;*

# The $p$-divisor

The main property of the $p$-divisor is the following.

**Proposition**

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ be an algebraic curve.*

- *If $C$ is $\mathcal{F}$-invariant then $\operatorname{ord}_C(\Delta_{\mathcal{F}}) > 0$;*
- *If $\operatorname{ord}_C(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$ then $C$ is $\mathcal{F}$-invariant.*

[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

# The $p$-divisor

The main property of the $p$-divisor is the following.

**Proposition**

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ be an algebraic curve.*

- *If $C$ is $\mathcal{F}$-invariant then $\mathrm{ord}_C(\Delta_{\mathcal{F}}) > 0$;*
- *If $\mathrm{ord}_C(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$ then $C$ is $\mathcal{F}$-invariant.*

[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

**Corollary**

*On the projective plane over characteristic $p > 0$ any foliation of degree $d$ such that $p \nmid d + 2$ has an invariant algebraic curve.*

# The $p$-divisor

The main property of the $p$-divisor is the following.

**Proposition**

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ be an algebraic curve.*

- *If $C$ is $\mathcal{F}$-invariant then $\mathrm{ord}_C(\Delta_{\mathcal{F}}) > 0$;*
- *If $\mathrm{ord}_C(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$ then $C$ is $\mathcal{F}$-invariant.*

[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

**Corollary**

*On the projective plane over characteristic $p > 0$ any foliation of degree $d$ such that $p \nmid d + 2$ has an invariant algebraic curve.*

**Proposition (J.V.Pereira)**

[a] *Let $\mathcal{F}$ be a foliation $\mathbb{P}^2_K$ and suppose that $\deg(\mathcal{F}) < p - 1$. Then, $\mathcal{F}$ has an invariant algebraic curve.*

[a] J.V.Pereira - **Invariant Hypersurfaces for Positive Characteristic Vector Fields**

# The $p$-divisor

**Corollary**

*On the projective plane over characteristic $p > 0$ any non-p-closed foliation of degree $d$ has an invariant algebraic curve of degree less than or equal to $p(d-1) + d + 2$.*

# The $p$-divisor

**Corollary**

*On the projective plane over characteristic $p > 0$ any non-p-closed foliation of degree $d$ has an invariant algebraic curve of degree less than or equal to $p(d-1)+d+2$.*

**Problem:** Let $\mathcal{F}$ be a foliation in the projective plane over the characteristic $p > 0$. How many solutions can $\mathcal{F}$ have?

# The $p$-divisor

**Corollary**

*On the projective plane over characteristic $p > 0$ any non-p-closed foliation of degree $d$ has an invariant algebraic curve of degree less than or equal to $p(d-1) + d + 2$.*

**Problem:** Let $\mathcal{F}$ be a foliation in the projective plane over the characteristic $p > 0$. How many solutions can $\mathcal{F}$ have?

**Proposition**

[a] *A foliation is p-closed if and only if it has infinitely many solutions.*

[a] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

# Curious examples

**Example**

*Suppose that $K$ has characteristic $3$ and consider the Jouanolou foliation of degree $2$ defined over $K$. Then, the $3$-divisor is irreducible with $\mathrm{sing}(\Delta_{\mathcal{F}}) \not\subset \mathrm{sing}(\mathcal{F})$.*

**Example (F.Touzet)**

*Consider the foliation $5$-closed foliation given by:*

$$\omega = 2z(x+y)dx + z(2z+x)dy + 4(2x^2 + 3xy + 2yz)dz$$

*The curve $C = \{-x^4 + x^3y + x^2yz + y^2z^2\}$ is $\mathcal{F}$-invariant with $[0:0:1] \in \mathrm{sing}(C)$ but not in $\mathrm{sing}(\mathcal{F})$.*

# Reduction modulo $p$

Consider the case where $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d$ defined by the projective 1-form:

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1}$$

# Reduction modulo $p$

Consider the case where $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d$ defined by the projective 1-form:

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1}$$

and let $\mathbb{Z}[\mathcal{F}]$ the finitely generated $\mathbb{Z}$-algebra obtained by adjoining all coefficients and their inverses that occur in $A, B$ and $C$

# Reduction modulo $p$

Consider the case where $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d$ defined by the projective 1-form:

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1}$$

and let $\mathbb{Z}[\mathcal{F}]$ the finitely generated $\mathbb{Z}$-algebra obtained by adjoining all coefficients and their inverses that occur in $A, B$ and $C$

**Example**

*Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ given by the 1-form:*

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

*for some $\alpha \in \mathbb{C} - \mathbb{Q}$. Then, the algebra associated is $\mathbb{Z}[\alpha, \alpha^{-1}]$*

# Reduction modulo $p$

Consider the case where $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d$ defined by the projective 1-form:

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1}$$

and let $\mathbb{Z}[\mathcal{F}]$ the finitely generated $\mathbb{Z}$-algebra obtained by adjoining all coefficients and their inverses that occur in $A, B$ and $C$

**Example**

*Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ given by the 1-form:*

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

*for some $\alpha \in \mathbb{C} - \mathbb{Q}$. Then, the algebra associated is $\mathbb{Z}[\alpha, \alpha^{-1}]$*

For the Jouanolou foliation, $A, B, C \in \mathbb{Z}[x, y, z]$ so that $\mathbb{Z}[\mathcal{F}_d] = \mathbb{Z}$.

# Reduction modulo $p$

**Fact:** For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$ the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

# Reduction modulo $p$

**Fact:** For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$ the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

Denote by $\omega_{\mathfrak{p}}$ the 1-form over $\overline{\mathbb{F}}_{\mathfrak{p}}$ obtained by reduction modulo $\mathfrak{p}$ of all coefficient which appears in $A, B$ and $C$. We obtain a non-zero element of $\mathrm{H}^0(\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \Omega^1_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}}(d+2))$ and $\omega_{\mathfrak{p}}$ determines a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ :

$$\omega_{\mathfrak{p}} = Adx + Bdy + Cdz \mod \mathfrak{p}$$

# Reduction modulo $p$

**Fact:** For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$ the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

Denote by $\omega_{\mathfrak{p}}$ the 1-form over $\overline{\mathbb{F}}_{\mathfrak{p}}$ obtained by reduction modulo $\mathfrak{p}$ of all coefficient which appears in $A, B$ and $C$. We obtain a non-zero element of $\mathrm{H}^0(\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \Omega^1_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}}(d+2))$ and $\omega_{\mathfrak{p}}$ determines a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ :

$$\omega_{\mathfrak{p}} = Adx + Bdy + Cdz \mod \mathfrak{p}$$

**Definition**

*The foliation determined by $\omega_{\mathfrak{p}}$ is denoted by $\mathcal{F}_{\mathfrak{p}}$ and is called the **reduction modulo** $p$ of $\mathcal{F}$.*

# Reduction modulo $p$

Natural question:

# Reduction modulo $p$

Natural question:

**Problem**

*Suppose that an abstract property* $\boldsymbol{P}$ *holds for* $\mathcal{F}_{\mathfrak{p}}$ *for an infinitely many primes (or almost all primes)* $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$*. What we can say about* $\mathcal{F}$*?*

# Reduction modulo $p$

Natural question:

**Problem**

*Suppose that an abstract property* $\boldsymbol{P}$ *holds for* $\mathcal{F}_{\mathfrak{p}}$ *for an infinitely many primes (or almost all primes)* $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$. *What we can say about* $\mathcal{F}$*?*

- **infinitely many primes** = primes in some dense subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$;
- **almost all primes** = primes in some non-empty open subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$.

# Reduction modulo $p$

Natural question:

**Problem**

*Suppose that an abstract property* $\boldsymbol{P}$ *holds for* $\mathcal{F}_{\mathfrak{p}}$ *for an infinitely many primes (or almost all primes)* $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$*. What we can say about* $\mathcal{F}$*?*

- **infinitely many primes** = primes in some dense subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$;
- **almost all primes** = primes in some non-empty open subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$.

When $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}$ then the notions: **infinitely many primes** and **all most primes** are the usual notions.

# Reduction modulo $p$

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;

# Reduction modulo $p$

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ is $p$-closed;

# Reduction modulo $p$

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ is $p$-closed;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ has irreducible/reduced $p$-divisor;

# Reduction modulo $p$

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ is $p$-closed;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ has irreducible/reduced $p$-divisor;

**Proposition**

*Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ and suppose that $\mathcal{F}_{\mathfrak{p}}$ has an invariant curve of degree less than $\boldsymbol{d}$ for almost all primes $\mathfrak{p}$. Then, $\mathcal{F}$ has an invariant curve of degree less than $\boldsymbol{d}$.*

# Reduction modulo $p$

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ is $p$-closed;
- the foliation $\mathcal{F}_{\mathfrak{p}}$ has irreducible/reduced $p$-divisor;

**Proposition**

*Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ and suppose that $\mathcal{F}_{\mathfrak{p}}$ has an invariant curve of degree less than $\boldsymbol{d}$ for almost all primes $\mathfrak{p}$. Then, $\mathcal{F}$ has an invariant curve of degree less than $\boldsymbol{d}$.*

**Idea:** the set $S(\mathcal{F}, K, d)$ of foliations on $\mathbb{P}^2_K$ that have invariant curves of degree $\leq d$ is algebraic variety over $K$. In particular, $S(\mathcal{F}, \mathbb{C}, d) \neq \varnothing$ if and only if $S(\mathcal{F}, \overline{\mathbb{F}}_{\mathfrak{p}}, d) \neq \varnothing$ for almost all primes $\mathfrak{p}$.

# Algebraic solutions

**Goal:** use reduction modulo $p$ to prove the non-algebraicity of holomorphic foliations

[1] Carnicer - **The Poincare problem in the nondicritical case**

# Algebraic solutions

**Goal:** use reduction modulo $p$ to prove the non-algebraicity of holomorphic foliations

**Proposition**

[a] *Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form $\omega = Adx + Bdy + Cdz$ with $A, B, C \in \mathbb{Z}[x, y, z]$. Let $p$ be a prime number such that $p > d + 2$. If $\Delta_{\mathcal{F}_p}$ is irreducible then $\mathcal{F}$ has no algebraic solutions.*

[a] W.Mendson - **Foliations on smooth algebraic surfaces in position characteristic**

[1] Carnicer - **The Poincare problem in the nondicritical case**

# Algebraic solutions

**Goal:** use reduction modulo $p$ to prove the non-algebraicity of holomorphic foliations

**Proposition**

[a] *Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form $\omega = Adx + Bdy + Cdz$ with $A, B, C \in \mathbb{Z}[x, y, z]$. Let $p$ be a prime number such that $p > d + 2$. If $\Delta_{\mathcal{F}_p}$ is irreducible then $\mathcal{F}$ has no algebraic solutions.*

[a] W.Mendson - **Foliations on smooth algebraic surfaces in position characteristic**

**Idea:** Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. We can assume that $F \in \mathbb{Z}[x, y, z]$. The Carnicer bound[1] implies that $\deg(C) \leq d + 2$. Reducing modulo $p$ and using the irreducibility of $\Delta_{\mathcal{F}_p}$ we get a contradiction.

[1] Carnicer - **The Poincare problem in the nondicritical case**

# Applications

**Corollary**

*The Jouanolou foliation of degree 2 or 3 has no algebraic solutions.*

# Applications

**Corollary**

*The Jouanolou foliation of degree 2 or 3 has no algebraic solutions.*

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$.

**Proposition**

*If the p-divisor $\Delta_{\mathcal{F}_{\mathfrak{p}}}$ is irreducible for almost all primes $\mathfrak{p}$ then $\mathcal{F}$ has no algebraic solutions.*

# Applications

**Corollary**

*The Jouanolou foliation of degree 2 or 3 has no algebraic solutions.*

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$.

**Proposition**

*If the $p$-divisor $\Delta_{\mathcal{F}_\mathfrak{p}}$ is irreducible for almost all primes $\mathfrak{p}$ then $\mathcal{F}$ has no algebraic solutions.*

**Idea:** Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. This curve has degree $e$. For large primes $p$ we obtain $C \mod p = \Delta_{\mathcal{F}_p}$, a contradiction since the degree of the $p$-divisor depends on $p$.

# From $\mathbb{F}_2$ to $\mathbb{C}$

Using similar ideas we can prove the following result (Jouanolou):

**Theorem**

*A very generic foliation of odd degree $d > 1$ with $d \not\equiv 1 \mod 3$ does not have algebraic solutions*

# From $\mathbb{F}_2$ to $\mathbb{C}$

Using similar ideas we can prove the following result (Jouanolou):

**Theorem**

*A very generic foliation of odd degree $d > 1$ with $d \not\equiv 1 \mod 3$ does not have algebraic solutions*

**Idea:** It is sufficient to show that the foliation $\mathcal{F}_d$ given by the projective 1-form

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

does not have any algebraic solutions.

# From $\mathbb{F}_2$ to $\mathbb{C}$

Using similar ideas we can prove the following result (Jouanolou):

**Theorem**

*A very generic foliation of odd degree $d > 1$ with $d \not\equiv 1 \mod 3$ does not have algebraic solutions*

**Idea:** It is sufficient to show that the foliation $\mathcal{F}_d$ given by the projective 1-form

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

does not have any algebraic solutions.

Suppose that there is an algebraic solution $C = \{F = 0\}$ given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$.

# From $\mathbb{F}_2$ to $\mathbb{C}$

Using similar ideas we can prove the following result (Jouanolou):

**Theorem**

*A very generic foliation of odd degree $d > 1$ with $d \not\equiv 1 \mod 3$ does not have algebraic solutions*

**Idea:** It is sufficient to show that the foliation $\mathcal{F}_d$ given by the projective 1-form

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

does not have any algebraic solutions.

Suppose that there is an algebraic solution $C = \{F = 0\}$ given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$.

By a technical lemma (using automorphism of $\mathcal{F}_d$) we can construct an irreducible curve $D = \{G = 0\}$ defined over $\mathcal{O}_L$ for some number field $L$ with $\deg(D) = d + 2 \equiv 1 \mod 2$.

## From $\mathbb{F}_2$ to $\mathbb{C}$

Using similar ideas we can prove the following result (Jouanolou):

**Theorem**

*A very generic foliation of odd degree $d > 1$ with $d \not\equiv 1 \mod 3$ does not have algebraic solutions*

**Idea:** It is sufficient to show that the foliation $\mathcal{F}_d$ given by the projective 1-form

$$\mathcal{F}_d\colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

does not have any algebraic solutions.

Suppose that there is an algebraic solution $C = \{F = 0\}$ given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$.

By a technical lemma (using automorphism of $\mathcal{F}_d$) we can construct an irreducible curve $D = \{G = 0\}$ defined over $\mathcal{O}_L$ for some number field $L$ with $\deg(D) = d + 2 \equiv 1 \mod 2$.

The reduction modulo 2 and comparison of degrees (the 2-divisor is irreducible) gives us a contradiction

# Jouanolou foliation in positive characteristic

**Theorem**

*Let $p > 2$ be a prime number such that $7 \nmid p + 4$ and such that $p \not\equiv 1 \mod 3$. Then, the Jouanolou foliation of degree two $\mathcal{F}_2$ defined over a field of characteristic $p$ has irreducible $p$-divisor.*

# Jouanolou foliation in positive characteristic

**Theorem**

*Let $p > 2$ be a prime number such that $7 \nmid p + 4$ and such that $p \not\equiv 1 \mod 3$. Then, the Jouanolou foliation of degree two $\mathcal{F}_2$ defined over a field of characteristic $p$ has irreducible $p$-divisor.*

In the proof, we use a particular automorphism of $\mathcal{F}_2$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ defined by a projective 1-form $\omega$.

# Jouanolou foliation in positive characteristic

**Theorem**

*Let $p > 2$ be a prime number such that $7 \nmid p + 4$ and such that $p \not\equiv 1 \mod 3$. Then, the Jouanolou foliation of degree two $\mathcal{F}_2$ defined over a field of characteristic $p$ has irreducible p-divisor.*

In the proof, we use a particular automorphism of $\mathcal{F}_2$.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ defined by a projective 1-form $\omega$.

An automorphism of a foliation $\mathcal{F}$ is a automorphism $\Phi$ of $\mathbb{P}^2_K$ which preserves $\mathcal{F}$:

$$\Phi^*\omega = \sigma(\Phi)\omega$$

for some $\sigma(\Phi) \in K^*$. The automorphism group of $\mathcal{F}$ is denoted by $\mathrm{Aut}(\mathcal{F})$.

# Jouanolou foliation: example

Let $\mathcal{F}_d$ defined by

$$\Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz.$$

[2] here we assume that $p$ does not divide $d^2 + d + 1$

# Jouanolou foliation: example

Let $\mathcal{F}_d$ defined by

$$\Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz.$$

Let $\gamma$ be a primitive $(d^2 + d + 1)$th root[2] of unity and consider

$$\Phi\colon \mathbb{P}^2_K \longrightarrow \mathbb{P}^2_K \qquad [x : y : z] \mapsto [\gamma^{d^2+1} x : \gamma y : z]$$

[2] here we assume that $p$ does not divide $d^2 + d + 1$

# Jouanolou foliation: example

Let $\mathcal{F}_d$ defined by

$$\Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz.$$

Let $\gamma$ be a primitive $(d^2 + d + 1)$th root[2] of unity and consider

$$\Phi\colon \mathbb{P}^2_K \longrightarrow \mathbb{P}^2_K \qquad [x : y : z] \mapsto [\gamma^{d^2+1} x : \gamma y : z]$$

Then $\Phi \in \mathrm{Aut}(\mathcal{F})$ has order $d^2 + d + 1$ with $\Phi^*\Omega = \gamma\Omega$.

[2] here we assume that $p$ does not divide $d^2 + d + 1$

# Jouanolou foliation: example

Let $\mathcal{F}_d$ defined by

$$\Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz.$$

Let $\gamma$ be a primitive $(d^2 + d + 1)$th root[2] of unity and consider

$$\Phi\colon \mathbb{P}^2_K \longrightarrow \mathbb{P}^2_K \qquad [x : y : z] \mapsto [\gamma^{d^2+1} x : \gamma y : z]$$

Then $\Phi \in \mathrm{Aut}(\mathcal{F})$ has order $d^2 + d + 1$ with $\Phi^*\Omega = \gamma\Omega$.

**Proposition**

*Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ non-p-closed and $\Phi \in \mathrm{Aut}(\mathcal{F})$. Then $\Phi^*\Delta_{\mathcal{F}} = \Delta_{\mathcal{F}}$.*

[2] here we assume that $p$ does not divide $d^2 + d + 1$

# Case $d = 2$

$$\mathcal{F}_2 \colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the $p$-divisor of $\mathcal{J} := \mathcal{F}_2$:

# Case $d = 2$

$$\mathcal{F}_2 \colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the $p$-divisor of $\mathcal{J} := \mathcal{F}_2$:

- The conditions $7 \nmid p + 4$ and $p \not\equiv 1 \mod 3$ imply that $\Delta_{\mathcal{J}} \neq 0$. Assume that there is an $\mathcal{J}$-invariant curve $C = \{F = 0\}$ with $\deg(C) < \deg(\Delta_{\mathcal{J}})$;

# Case $d = 2$

$$\mathcal{F}_2 \colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the $p$-divisor of $\mathcal{J} := \mathcal{F}_2$:

- The conditions $7 \nmid p + 4$ and $p \not\equiv 1 \mod 3$ imply that $\Delta_{\mathcal{J}} \neq 0$. Assume that there is an $\mathcal{J}$-invariant curve $C = \{F = 0\}$ with $\deg(C) < \deg(\Delta_{\mathcal{J}})$;
- We can write:
$$dF \wedge \omega = F\left(\left(\frac{\deg(C)}{4}\right) d\omega + i_R \beta\right)$$
with $\beta \neq 0$ a homogeneous 3-form on $\mathbb{A}^3_K$ of degree 1;

# Case $d = 2$

$$\mathcal{F}_2\colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the $p$-divisor of $\mathcal{J} := \mathcal{F}_2$:

- The conditions $7 \nmid p + 4$ and $p \not\equiv 1 \mod 3$ imply that $\Delta_{\mathcal{J}} \neq 0$. Assume that there is an $\mathcal{J}$-invariant curve $C = \{F = 0\}$ with $\deg(C) < \deg(\Delta_{\mathcal{J}})$;
- We can write:
$$dF \wedge \omega = F\left(\left(\frac{\deg(C)}{4}\right)d\omega + i_R\beta\right)$$
with $\beta \neq 0$ a homogeneous 3-form on $\mathbb{A}^3_K$ of degree 1;
- We have two cases:
  1. some power $\Phi^l$ fixes the curve $C$: computation implies that $\beta = 0$, contradiction;

# Case $d = 2$

$$\mathcal{F}_2 \colon \Omega_2 = (x^d z - y^3)dx + (xy^2 - z^3)dy + (z^2 y - x^3)dz.$$

Idea of proof of the irreducibility of the $p$-divisor of $\mathcal{J} := \mathcal{F}_2$:

- The conditions $7 \nmid p + 4$ and $p \not\equiv 1 \mod 3$ imply that $\Delta_{\mathcal{J}} \neq 0$. Assume that there is an $\mathcal{J}$-invariant curve $C = \{F = 0\}$ with $\deg(C) < \deg(\Delta_{\mathcal{J}})$;
- We can write:

$$dF \wedge \omega = F\left(\left(\frac{\deg(C)}{4}\right) d\omega + i_R \beta\right)$$

  with $\beta \neq 0$ a homogeneous 3-form on $\mathbb{A}^3_K$ of degree 1;
- We have two cases:
  1. some power $\Phi^l$ fixes the curve $C$: computation implies that $\beta = 0$, contradiction;
  2. $\Phi^l$ does not fix $C$ for all $l \in \{1, \ldots, 6\}$: In that case for any prime divisor $P$ in the support of $\Delta_{\mathcal{J}}$ we have: $P, \Phi^* P, \ldots, (\Phi^6)^* P$ are disticts and this implies that $7 \mid p + 4$, a contradiction;

# Jouanolou foliation

**Corollary**

*In the conditions of the Theorem, a generic foliation of degree two on the projective plane over characteristic $p > 0$ has irreducible $p$-divisor.*

- $\mathcal{F}_d$ on $\mathbb{P}^2_{\mathbb{C}}$:

$$\omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

1. $(p, d) = (5, 2)$ :

$$\Delta_{\mathcal{F}_{5,2}} = [i_{v_2^5} \omega_2] = \{X^5 Z^4 + X^4 Y^5 + 2X^3 Y^3 Z^3 + Y^4 Z^5 = 0\} \in \mathrm{Div}(\mathbb{P}^2_{\overline{\mathbb{F}}_5})$$

2. $(p, d) = (11, 3)$ :

$$\Delta_{\mathcal{F}_{11,3}} = [i_{v_3^{11}} \omega_3] = \{X^{19} Z^8 - 2X^{16} Y^4 Z^7 + \cdots + 3XY^{11} Z^{15} + Y^8 Z^{19} = 0\} \in \mathrm{Div}(\mathbb{P}^2_{\overline{\mathbb{F}}_{11}})$$

# Jouanolou foliation II

For $d > 2$ we have:

**Theorem**

[a] *Let $K$ be an algebraically closed field of characteristic $p > 0$. Let $d \in \mathbb{Z}_{>0}$ such that*

- $p < d$ *and* $p \not\equiv 1 \mod 3$*;*
- $d^2 + d + 1$ *is prime.*

*Then the Jouanolou foliation $\mathcal{F}_d$ has irreducible p-divisor* ***or*** $\Delta_{\mathcal{F}_d} = C + pR$ *with* $\deg(C) = pl + d + 2$ *with $l > 0$ and $R$ not $\mathcal{F}_d$-invariant.*

[a] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**

# Jouanolou foliation II

For $d > 2$ we have:

**Theorem**

[a] *Let $K$ be an algebraically closed field of characteristic $p > 0$. Let $d \in \mathbb{Z}_{>0}$ such that*

- $p < d$ *and* $p \not\equiv 1 \mod 3$;
- $d^2 + d + 1$ *is prime.*

*Then the Jouanolou foliation $\mathcal{F}_d$ has irreducible p-divisor* ***or*** $\Delta_{\mathcal{F}_d} = C + pR$ *with* $\deg(C) = pl + d + 2$ *with* $l > 0$ *and $R$ not $\mathcal{F}_d$-invariant.*

[a] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**

**Consequence:** the Jouanolou foliation $\mathcal{F}_d$ has a unique algebraic curve.

# Jouanolou foliation II

For $d > 2$ we have:

**Theorem**

[a] *Let $K$ be an algebraically closed field of characteristic $p > 0$. Let $d \in \mathbb{Z}_{>0}$ such that*

- $p < d$ *and* $p \not\equiv 1 \mod 3$*;*
- $d^2 + d + 1$ *is prime.*

*Then the Jouanolou foliation $\mathcal{F}_d$ has irreducible p-divisor* ***or*** $\Delta_{\mathcal{F}_d} = C + pR$ *with* $\deg(C) = pl + d + 2$ *with $l > 0$ and $R$ not $\mathcal{F}_d$-invariant.*

[a] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**

**Consequence:** the Jouanolou foliation $\mathcal{F}_d$ has a unique algebraic curve.

**Remark**

[a] *It is conjectured that there are infinitely many $d \in \mathbb{Z}_{>0}$ such that $d^2 + d + 1$ is prime. This is a particular case of the Bunyakovsky conjecture (1857).*

[a] https://mathoverflow.net/questions/438807/primes-of-the-form-d2d1

# characteristic $p = 5$ and $d \leq 100$

| $d$ | $d^2 + d + 1$ | $\deg(C)$ | $l$ | $R$ |
|---|---|---|---|---|
| 6 | 43 | 18 | 2 | $\{xyz = 0\}$ |
| 12 | 157 | 39 | 5 | $\{xyz = 0\}$ |
| 17 | 307 | 54 | 7 | $\{xyz = 0\}$ |
| 21 | 463 | 63 | 8 | $\{xyz = 0\}$ |
| 27 | 757 | 84 | 11 | $\{xyz = 0\}$ |
| 41 | 1723 | 123 | 16 | $\{xyz = 0\}$ |
| 57 | 3307 | 174 | 23 | $\{xyz = 0\}$ |
| 62 | 3907 | 189 | 25 | $\{xyz = 0\}$ |
| 66 | 4423 | 198 | 26 | $\{xyz = 0\}$ |
| 71 | 5113 | 213 | 28 | $\{xyz = 0\}$ |
| 77 | 6007 | 234 | 31 | $\{xyz = 0\}$ |

- for $d \in \{2, 14, 24, 54, 59, 69, 89, 99\}$, $\Delta_{\mathcal{F}_d}$ is irreducible;
- for the remaining cases, $\Delta_{\mathcal{F}_d} = 0$.

Thank you ;-)