The space of foliations (in positive characteristic)

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Seminário de Geometria Algébrica e Geometria Complexa da UFF

June 04, 2025

Structure

- Part I: Introduction

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- Part II: Codimension-one case in small characteristics/degree

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- Part III: Special irreducible components

Part I: Introduction

Foliations in projective spaces

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K = algebraically closed field

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$$i_v\omega \wedge \omega = 0 \qquad \forall v \in \bigwedge^{q-1} \mathbb{A}^{n+1}_K \tag{2}$$

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- The q-form ω is **integrable**:
$$i_v\omega \wedge d\omega = 0 \qquad \forall v \in \bigwedge^{q-1} \mathbb{A}^{n+1}_K \tag{3}$$

Foliation in projective spaces

It follows from Euler's sequence that we can identify $\mathrm{H}^0(\mathbb{P}^n_K, \Omega^q_{\mathbb{P}^n_K}(d+q+1))$ with the vector space of homogeneous polynomial q-forms

$$\omega = \sum_{0 \leq i_1 < \cdots < i_q \leq n} a_I(x_0, \ldots, x_n) dx_I$$

which are annihilated by the radial vector field $R = \sum_{i=0} x_i \partial_{x_i}$

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- A **projective q-form of degree** $d+q+1$ is a homogeneous q-form in $\mathbb{A}^{n+1}_K$ representing an element of $\mathrm{H}^0(\mathbb{P}^n_K, \Omega^q_{\mathbb{P}^n_K}(d+q+1))$

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Example: Let $\alpha \in K^*$. The 1-form

$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz.$$

is a projective 1-form of degree 3 on $\mathbb{A}^3_K$ and defines a foliation of degree 1 on $\mathbb{P}^2_K$.

The space of foliation in projective spaces

The **space of codimension q foliations of degree d** on the projective space $\mathbb{P}^n_K$ is the quasi-projective variety of $\mathbb{P}\,\mathrm{H}^0(\mathbb{P}^n_K, \Omega^q_{\mathbb{P}^n_K}(d+q+1))$:

$$\mathsf{Fol}^q_d(\mathbb{P}^n_K) = \{[\omega] \in \mathbb{P}\,\mathrm{H}^0(\mathbb{P}^n_K, \Omega^q_{\mathbb{P}^n_K}(d+q+1)) \mid \omega \text{ satisfies 1, 2 and 3}\}$$

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For all integers $d \geq 0$, $1 \leq q \leq n-1$, and $n \geq 3$, describe the irreducible components of $\mathsf{Fol}^q_d(\mathbb{P}^n_{\mathbb{C}})$.

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- Note that $\mathsf{Fol}^1_d(\mathbb{P}^2_K)$ is irreducible, as the integrability condition holds automatically for dimensional reasons.

Codimension one foliations on projective spaces

A codimension one foliation of degree d on $\mathbb{P}^n_K$ is given by a homogeneous 1-form on the affine space $\mathbb{A}^{n+1}_K$

$$\sigma = A_0(x_0, \dots, x_n)dx_0 + \cdots + A_n(x_0, \dots, x_n)dx_n \in \mathrm{H}^0(\mathbb{P}^n_K, \Omega^1_{\mathbb{P}^n_K}(d+2))$$

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where $A_0 \dots, A_n \in K[x_0, \dots, x_n]$ are homogeneous of degree $d+1$ and such that $\mathrm{sing}(\sigma) = \mathcal{Z}(A_0 \dots, A_n)$ has codimension ≥ 2 and with σ having the following properties:

$$i_R\sigma = \sum_i A_i x_i = 0 \qquad \sigma \wedge d\sigma = 0.$$

Codimension one foliations on projective spaces

The integrability condition gives equations:

$$A_i \left(\frac{\partial A_l}{\partial x_j} - \frac{\partial A_j}{\partial x_l} \right) + A_j \left(\frac{\partial A_i}{\partial x_l} - \frac{\partial A_l}{\partial x_i} \right) + A_l \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) = 0$$

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The space of codimension one foliations of degree $d \geq 0$ on $\mathbb{P}^n_K$ $(n \geq 2)$ is

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Problem

Describe the irreducible components of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.

Some irreducible components of $\mathsf{Fol}^1_d(\mathbb{P}^n_K)$

Some known results:

- **Degree zero and one**: $\mathsf{Fol}^1_0(\mathbb{P}^n_{\mathbb{C}})$ is irreducible and identified with the Grassmannian of lines in $\mathbb{P}^n_{\mathbb{C}}$. When $d = 1$, the space $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathbb{C}})$ has exactly two irreducible components.[1][2]

[1] Alcides Lins Neto, **Irreducible components of the space of foliations**

[2] F. Loray, J. V. Pereira, and F. Touzet. **Foliations with trivial canonical bundle on Fano 3-folds**

[3] **Irreducible components of the space of holomorphic foliations of degree two in C**P(n)

[4] Maurício Corrêa and Alan Muniz **Holomorphic foliations of degree two and arbitrary dimension**

[5] **Codimension one foliations of degree three on projective spaces**

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- **Degree two**: For codimension one foliations of degree $d = 2$ on $\mathbb{P}^n_{\mathbb{C}}$, Cerveau and Lins Neto showed[3] that $\mathsf{Fol}^1_2(\mathbb{P}^n_{\mathbb{C}})$ has exactly six irreducible components and described them explicitly.[4]

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Some irreducible components of $\mathsf{Fol}_d^1(\mathbb{P}_K^n)$

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- **Degree zero and one**: $\mathsf{Fol}_0^1(\mathbb{P}_{\mathbb{C}}^n)$ is irreducible and identified with the Grassmannian of lines in $\mathbb{P}_{\mathbb{C}}^n$. When $d = 1$, the space $\mathsf{Fol}_1^1(\mathbb{P}_{\mathbb{C}}^n)$ has exactly two irreducible components.[1 2]

- **Degree two**: For codimension one foliations of degree $d = 2$ on $\mathbb{P}_{\mathbb{C}}^n$, Cerveau and Lins Neto showed[3] that $\mathsf{Fol}_2^1(\mathbb{P}_{\mathbb{C}}^n)$ has exactly six irreducible components and described them explicitly.[4]

- **Degree three**: In a recent work[5], R.C. da Costa, R. Lizarbe, and J.V. Pereira used a structure theorem to describe exactly 18 irreducible components of $\mathsf{Fol}_3^1(\mathbb{P}_{\mathbb{C}}^n)$ whose generic element admits no meromorphic first integral. They also show that $\mathsf{Fol}_3^1(\mathbb{P}_{\mathbb{C}}^n)$ has at least 24 distinct irreducible components.

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Components in degree $d \geq 3$

- **Rational components**: Let F, G be irreducible homogeneous polynomials of degrees p and q, respectively. Assume F and G are coprime and $d = p + q - 2$. Then, $\omega = qFdG - pGdF$ defines a foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree d. Let $\mathcal{R}(p,q)$ denote the set of such foliations. The closure $\overline{\mathcal{R}(p,q)}$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.[6][7][8]

[6] Gómez-Mont and A. Lins Neto — **Structural stability of foliations with a meromorphic first integral**

[7] F. Cukierman, J. V. Pereira, I. Vainsencher — **Stability of foliations induced by rational maps**

[8] W. Mendson and J. V. Pereira — **The space of foliations on projective spaces in positive characteristic**

[9] Cerveau, Lins Neto and Edixhoven — **Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$**

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- **Pullback components**: Let $\mathcal{G}$ be a codimension one foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree e, defined by a projective 1-form ω. Let $F\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ be a dominant rational map of degree m. Then $F^*\omega$ defines a foliation of degree $d = (e + 2)m - 2$ on $\mathbb{P}^n_{\mathbb{C}}$. Let $\mathrm{PB}(m, e, n)$ be the set of these foliations. Then $\overline{\mathrm{PB}(m, e, n)}$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.[9]

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[9] Cerveau, Lins Neto and Edixhoven — **Pull-back components of the space of holomorphic foliations on $\mathbb{CP}(n)$, $n \geq 3$**

Components in degree $d \geq 3$

- **Logarithmic components**: Let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r$ be homogeneous polynomials with $\deg(F_i) = d_i$. Assume $F_1, \ldots, F_r$ are irreducible and pairwise coprime. Let $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ such that $\sum_{i=1}^{r} \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

The 1-form Ω defines a foliation $\mathcal{F}_\Omega$ of codimension one and degree $d = \sum_i d_i - 2$ on $\mathbb{P}^n_{\mathbb{C}}$. In this case, we say that $\mathcal{F}_\Omega$ is a logarithmic foliation of type $(d_1, \ldots, d_r)$. Let $\mathrm{Log}_n(d_1, \ldots, d_r)$ denote the set of such foliations. Then the closure $\overline{\mathrm{Log}_n(d_1, \ldots, d_r)}$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.[10][11][12]

[10] O. Calvo-Andrade — **Irreducible components of the space of foliations**

[11] F. Cukierman, J. Gargiulo and C.D. Massri — **Stability of logarithmic differential one-forms**

[12] W. Mendson and J. V. Pereira —**The space of foliations in projective spaces in positive characteristic**

Irreducible components via reduction mod p

The technique of reduction modulo p for codimension one foliation on the projective spaces can be used to give a proof of the following theorem about irreducible components of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$:

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Theorem

[a][b] *Let $d \in \mathbb{Z}_{\geq 3}$ and $d_1, d_2 \in \mathbb{Z}_{>0}$ such that $d = d_1 + d_2 + 2$. Let $\mathrm{PBB}(d_1, d_2)$ be the set of foliations on $\mathbb{P}^n_{\mathbb{C}}$ that are linear pullback of a foliation of type (d_1, d_2) on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Then $\overline{\mathrm{PBB}(d_1, d_2)}$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.*

[a] W. Mendson - **Folheações de codimensão um em característica positiva e aplicações**
[b] W. Mendson, J. V. Pereira - **Codimension one foliations in positive characteristic**

The topics in the proof include:

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The topics in the proof include:

- the struture of the p-divisor for generic foliations on $\mathbb{P}^1_K \times \mathbb{P}^1_K$;

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The topics in the proof include:

- the struture of the p-divisor for generic foliations on $\mathbb{P}^1_K \times \mathbb{P}^1_K$;
- proving the analogous theorem in positive characteristic and lift to characteristic 0.

Part II: Codimension one in small characteristics/degree

Technical lemma

We will make use of Medeiros' description of linear, locally decomposable, and integrable differential forms. The original statement was formulated over the field $\mathbb{C}$, but the result remains valid over any algebraically closed field K.

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Moreover, if ω is integrable, then in Case 1 $\alpha = df$ for some quadratic polynomial $f \in K[x_0, \ldots, x_n]$ and, in Case 2, the polynomials A_i belong to $K[x_0, \ldots, x_q]$.

[a] A. S de Medeiros — **Structural stability of integrable differential forms**

[b] W. Mendson and J. V. Pereira — **The space of foliations on projective spaces in positive characteristic**

Degree zero

Theorem

[a] *Let K be an algebraically closed field. Let $n \geq 3$ and $1 \leq q \leq n-1$ be integers.*

- *If the characteristic of K is different from 2, or if $q > 1$, then foliations of degree zero are defined by linear projections $\mathbb{P}^n_K \dashrightarrow \mathbb{P}^q_K$.*
- *If the characteristic of K is 2 and $q = 1$, then $\mathsf{Fol}^1_0(\mathbb{P}^n_K) = \mathbb{P}H^0(\mathbb{P}^n_K, \Omega^1_{\mathbb{P}^n_K}(2))$.*

In all cases, $\mathsf{Fol}^q_0(\mathbb{P}^n_K)$ is an irreducible algebraic variety.

[a] W. Mendson and J. V. Pereira — **The space of foliations on projective spaces in positive characteristic**

Degree zero

Theorem

[a] *Let K be an algebraically closed field. Let $n \geq 3$ and $1 \leq q \leq n-1$ be integers.*

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In all cases, $\mathsf{Fol}^q_0(\mathbb{P}^n_K)$ is an irreducible algebraic variety.

[a] W. Mendson and J. V. Pereira — **The space of foliations on projective spaces in positive characteristic**

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- The second case corresponds to the **linear pullback:**

$$\omega = \sum_{i=0}^{q} A_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q \, .$$

with $A_i \in K[x_0, \ldots, x_q]$.

Some irreducible components in characteristic $p > 0$

Proposition

[a] *Let* K *be an algebraically closed field of characteristic* $p > 0$*. For every integer* $n \geq 3$ *and* $e \geq 1$*, there exists an irreducible component* $Cl_{pe-2}(\mathbb{P}^n_K)$ *of* $\mathsf{Fol}^1_{pe-2}(\mathbb{P}^n_K)$ *given by an open subset of the projectivization of the vector space of closed polynomial projective* 1*-forms of degree* pe *on* $\mathbb{A}^{n+1}_K$*.*

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[13] K. Saito **On a generalization of de Rham lemma**

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[13] K. Saito **On a generalization of de Rham lemma**

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[a] *Let K be an algebraically closed field of characteristic $p \geq 0$.*

1. *If $p = 2$ then $\mathsf{Fol}_1^1(\mathbb{P}^n_K)$ has exactly one irreducible component: $\overline{\mathrm{PB}(1,1,n)}$.*
2. *If $p = 3$ then $\mathsf{Fol}_1^1(\mathbb{P}^n_K)$ has exactly two irreducible components: $\overline{\mathrm{PB}(1,1,n)}$ and $Cl_1(\mathbb{P}^n_K)$.*
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- **Case 1**: $d\omega = df \wedge dx_0$ for some quadratic function $f \in K[x_0, \ldots, x_n]$, or
- **Case 2**: $d\omega = \sum_{i,j \in \{0,1,2\}} A_{ij} dx_i \wedge dx_j$ for some constants $A_{ij} \in K$.

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- If $p = 2$ and $d\omega$ is as in Case 1, then

$$\omega = 3\omega = i_R(d\omega) = 2f dx_0 + x_0 df = x_0 df \, .$$

This implies that ω is either identically zero or that ω has a zero locus of codimension one. In either case, ω does not define a codimension foliation of degree one. Thus, in characteristic two, the only irreducible component of $\mathsf{Fol}_1^1(\mathbb{P}^n_K)$ is $\overline{\mathrm{PB}(1,1,n)}$.

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Part III: Special irreducible components

Special foliations in codimension one

Fix K of characteristic $p > 0$ and let $F \in \mathrm{H}^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(p))$. Note that dF defines an element of $\mathrm{H}^0(\mathbb{P}^n_K, \Omega^1_{\mathbb{P}^n_K}(p)$ and thus defines a foliation if $\operatorname{codim} \operatorname{sing}(dF) \geq 2$.

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Let $Cl_{p-2}(\mathbb{P}^n_K)$ be the Zariski closure of the image Φ. Then, $Cl_{p-2}(\mathbb{P}^n_K)$ is an irreducible component of $\mathsf{Fol}^1_{p-2}(\mathbb{P}^n_K)$.

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Lemma

[a b] *Let $X \subset \mathsf{Fol}^q_d(\mathbb{P}^n_K)$ be an irreducible subvariety. If there exists $[\omega] \in X$ such that the Zariski tangent space of $\mathsf{Fol}^q_d(\mathbb{P}^n_K)$ has dimension equal to the dimension of X then X is an irreducible component of $\mathsf{Fol}^q_d(\mathbb{P}^n_K)$.*

[a] R. C. da Costa, R. Lizarbe and J. V. Pereira — **Codimension one foliations of degree three on projective spaces**

[b] Andrew Hubery — **Irreducible components of quiver Grassmannians,**

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[14] K. Saito **On a generalization of de Rham lemma**

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$$F = x_0 x_1^{p-1} + x_1 x_2^{p-1} + \cdots + x_{n-1} x_n^{p-1} + x_0^p$$

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Lemma

Let $\alpha \in \mathrm{H}^0(\mathbb{P}^n_K, \Omega^1_{\mathbb{P}^n_K}(p))$ be a closed projective 1-form. Then there exists a homogeneous polynomial G of degree p, such that $\alpha = dG$.

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Proposition

Let K be an algebraically closed field of characteristic $p > 0$. For every integer $n \geq 3$ and $e \geq 1$, there exists an irreducible component $Cl_{pe-2}(\mathbb{P}^n_K)$ of $\mathsf{Fol}^1_{pe-2}(\mathbb{P}^n_K)$ given by an open subset of the projectivization of the vector space of closed polynomial projective 1-forms of degree pe on $\mathbb{A}^{n+1}_K$.

[15] F. Cukierman, J. V. Pereira, I. Vainsencher — **Stability of foliations induced by rational maps**

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Theorem

[a] ***Assume*** $1 \leq q < n-1$. *Consider the rational map*

$$\begin{array}{rcl} \Phi : \mathbb{P}\mathbf{S}_p \times \cdots \times \mathbb{P}\mathbf{S}_p & \dashrightarrow & \mathsf{Fol}^q_d(\mathbb{P}^n_K) \\ (F_1, \ldots, F_q) & \mapsto & dF_1 \wedge \cdots \wedge dF_q \end{array}$$

where $d = qp - q \underline{- 1.}$ Let U be the largest open subset where Φ is a morphism. Then the closure $\overline{\Phi(U)}$ of $\Phi(U)$ in is an irreducible component of $\mathsf{Fol}^q_d(\mathbb{P}^n_K)$.

[a] T. Fassarella, J. P. Figueredo and W. Mendson — **The space of p-closed foliations on projective spaces (in progress)**

[15] F. Cukierman, J. V. Pereira, I. Vainsencher — **Stability of foliations induced by rational maps**

Idea of proof

- Let $\underline{F} = (F_1, \ldots, F_q) \in \mathbb{P}\mathbf{S}_p \times \cdots \times \mathbb{P}\mathbf{S}_p$ such that the projective j-form
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- By multilinearity, the map induced in tangent spaces is given by

$$d\Phi_{\underline{F}}(\underline{G}) = \sum_{j=1}^{q} dF_1 \wedge \cdots \wedge dF_{j-1} \wedge dG_j \wedge dF_{j+1} \wedge \cdots \wedge dF_q.$$

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$$dF_i \wedge dF_j \wedge \eta = 0 \qquad \forall i, j$$
- The last formula implies that there are homogeneous 1-forms α_i, $i = 1, \ldots, q$, such that
$$\eta = \sum_{j=1}^{q} dF_1 \wedge \cdots \wedge dF_{j-1} \wedge \alpha_j \wedge dF_{j+1} \wedge \cdots \wedge dF_q.$$
From this and $i_R\eta = 0$, we see that $i_R\alpha_j = 0$ for every $j = 1, \ldots, q$. Then $\alpha_j \in \mathrm{H}^0(\mathbb{P}^n_K, \Omega^1_{\mathbb{P}^n_K}(p))$, $j = 1, \ldots, q$.

- Since $d\eta = 0$, differentiating both sides yields

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- Comparing degrees, we see that all α_{ji} vanish identically. Hence $d\alpha_j = 0$ for every $j = 1, \ldots, q$. It follows that there are homogeneous polynomials G_j of degree p such that $\alpha_j = dG_j$.

Why is not $q = n - 1$ included?

In the case $q = n - 1$, the integrability is automatic, but we can consider **the space of p-closed foliations**, $\mathrm{P}\mathsf{Fol}_d^{n-1}(\mathbb{P}_K^n)$, and study the problem of irreducible components in this space.

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- Consider the rational map

$$\begin{aligned}\Phi\colon \mathbb{P}\mathbf{S}_p \times \cdots \times \mathbb{P}\mathbf{S}_p &\dashrightarrow \mathrm{PFol}_d^{n-1}(\mathbb{P}_K^n)\\ (F_1, \ldots, F_{n-1}) &\mapsto dF_1 \wedge \cdots \wedge dF_{n-1}.\end{aligned}$$

  Let U be the largest open subset where Φ is a morphism.

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Question: Is the closure $\overline{\Phi(U)}$ of $\Phi(U)$ in $\mathsf{PFol}_d^{n-1}(\mathbb{P}_K^n)$ an irreducible component of $\mathsf{PFol}_d^{n-1}(\mathbb{P}_K^n)$?

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Remark

So far, we do not know if there is an example of a foliation of the type $dF_1 \wedge \cdots \wedge dF_{n-1}$ in characteristic $p > 2$, which has a singular set of ***codimension at least*** *3. This prevents the division lemma from working, and it is not possible to apply the same technical lemmas as applied in case $q < n - 1$. Jason Starr provides answers in characteristic two on MathOverflow.*

About differential forms in positive characteristic

Asked 2 months ago Modified 2 months ago Viewed 575 times

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Fix K to be an algebraically closed field of characteristic $p > 0$. I would like to know if there exists an example of $F_1, \dots, F_{n-1} \in K[x_0, \dots, x_n]_p$ (homogeneous of degree p) such that the variety defined by the locus of $dF_1 \wedge \cdots \wedge dF_{n-1}$ in $\mathbb{P}^n_K$ $(n > 2)$ has codimension at least 3. I appreciate any references or comments about this.

I asked this here but I did not get answers.

ag.algebraic-geometry ac.commutative-algebra characteristic-p

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asked Mar 23 at 15:01

numberwat

Thank you ;-)