Notes on unimodular domains

Remember the following inequality (cf.[1][Corollary 2.7])

Lemma 1. *Let* $f \in \mathbb{F}_p[X_1, ..., X_n]$ *be a non-constant polynomial and define* $X = Spec(\mathbb{F}_p[X_1, ..., X_n]/(f))$. *Then,*

$$\#X(\mathbb{F}_p) \leq \deg(f)p^{n-1}.$$

The following proposition refines many of results that are contained on my notes "On unimodular and invariant domains"

Proposition 1. *Let* $p \in \mathbb{Z}_{>0}$ *be a prime number and consider the* p*-adic ring* $\mathbb{Z}_p$. *Let* $f = (f_1, ..., f_n) : \mathbb{Z}_p^n \longrightarrow \mathbb{Z}_p^n$ *be a polynomial map with* $\det J_f = 1$. *If* $deg(f_j) < p$ ***for some*** j *then* $\#X_1(\mathbb{Z}_p) < p^n$, *where*

$$X_1 = Spec(\mathbb{Z}_p[X_1, ..., X_n]/(f_1, \cdots, f_n))$$

Proof. By normalization we can assume that $f_i \neq 0 \mod p$ for every $i \in \{1, \cdots, n\}$. Define X_2 by

$$X_2 = Spec(\mathbb{F}_p[X_1, ..., X_n]/(\overline{f_1}, \cdots, \overline{f_n}))$$

The condition $\det J_f = 1$ and the Hensel lemma implies

$$\#X_1(\mathbb{Z}_p) = \#X_2(\mathbb{F}_p).$$

If we define $X_{f_j} = Spec(\mathbb{F}_p[X_1, ..., X_n]/(\overline{f_j}))$ we have by the lemma above that

$$\#X(\mathbb{Z}_p) = \#X(\mathbb{F}_p) \leq \#X_f(\mathbb{F}_p) \leq \deg(f)p^{n-1} < p^n$$

□

References

[1] S.R. Ghorpade. A note on Nullstellensatz over finite fields. https://arxiv.org/abs/1806.09489v2.