On reduction modulo p of foliations

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Structure

- Part I: Introduction;
- Part II: Reduction modulo p;
- Part III: Applications to foliations over $\mathbb{C}$.

Part I: Introduction

Foliations

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A **foliation**, $\mathcal{F}$, of degree d on the projective plane $\mathbb{P}^2_K$ is given, mod K^*, by a non-zero element $\omega \in \mathrm{H}^0(\mathbb{P}^2_K, \Omega^1_{\mathbb{P}^2_K}(d+2))$ with finite singular locus

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Explicitly:

- Using the Euler exact sequence we can see ω as a projective 1-form:

$$\omega = Adx + Bdy + Cdz$$

on $\mathbb{A}^3_K$ such that $A, B, C \in K[x, y, z]$ are homogeneous of degree $d+1$ and $Ax + By + Cz = 0$ with

$$\mathrm{sing}(\omega) = \mathcal{Z}(A, B, C) = \{p \in \mathbb{P}^2_K \mid A(p) = B(p) = C(p) = 0\}$$

finite.

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- In this sense, a foliation of degree d on $\mathbb{P}^2_K$ is determined, modulo K^*, by a homogeneous vector field on $\mathbb{A}^3_K$:

$$v = A_0\partial_x + A_1\partial_y + A_2\partial_z \in \mathfrak{X}_d(\mathbb{A}^3_K)$$

where $A_0, A_1, A_2 \in K[x,y,z]$ are homogeneous of degree d with

$$\mathbf{div}(v) = \partial_x A_0 + \partial_y A_1 + \partial_z A_2 = 0$$

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The equivalence of these notions is given by the following result:

Proposition

[a] *There is a bijection between the set of projective 1-forms on $\mathbb{A}^3_K$ of degree $d+1$ and homogeneous vector fields with divergent zero of degree d.*

[a] Jouanolou - **Equations de Pfaff algébriques**

Example

Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form

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and write

$$d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy).$$

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Then ω defines a foliation of degree 1 on $\mathbb{P}^2_K$ and the vector field associated is given by:

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2-\alpha}{3}\right) y\partial_y + \left(\frac{-1-\alpha}{3}\right) z\partial_z$$

Invariant curves

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*The curve C is **$\mathcal{F}$-invariant** if there is a homogeneous 2-form σ on $\mathbb{A}^3_K$ such that*

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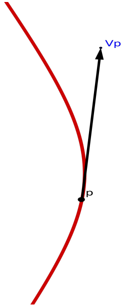

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- **Logarithmic foliations**: Let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r \in K[x, y, z]$ homogeneous polynomials with $d_i = \deg(F_i)$. Suppose that $F_1, \ldots, F_r$ are irreducible and coprime. Let $\alpha_1, \ldots, \alpha_r \in K^*$ such that $\sum_{i=1}^r \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_{r-1} F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

The 1-form Ω defines, $\mathcal{F}_\Omega$, a foliation of degree $d = \sum_i d_i - 2$ on $\mathbb{P}^2_K$. We say that $\mathcal{F}_\Omega$ is a **logarithmic foliation** of type $(d_1, \ldots, d_r)$. The curves $C_i = \{F_i = 0\}$ are $\mathcal{F}_\Omega$-invariant.

Jouanolou example: foliations without invariant curves

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the projective 1-form:

$$\mathcal{F}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

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Theorem (Jouanolou)

[a] *If $K = \mathbb{C}$ the foliation $\mathcal{F}_d$ does not have invariant algebraic curves.*

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This result implies in particular that on $\mathbb{P}^2_{\mathbb{C}}$ **almost all** foliation on the complex projective plane have no algebraic invariant curves.

Part II: Reduction modulo p

Reduction modulo p

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For the Jouanolou foliation, $A, B, C \in \mathbb{Z}[x, y, z]$ so that $\mathbb{Z}[\mathcal{F}_d] = \mathbb{Z}$.

Reduction modulo p

Fact: For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$ the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

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Denote by $\omega_{\mathfrak{p}}$ the 1-form over $\overline{\mathbb{F}}_{\mathfrak{p}}$ obtained by reduction modulo $\mathfrak{p}$ of all coefficient which appears in A, B and C. We obtain a non-zero element of $\mathrm{H}^0(\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \Omega^1_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}}(d+2))$ and $\omega_{\mathfrak{p}}$ determines a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}$:

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Definition

*The foliation determined by $\omega_{\mathfrak{p}}$ is denoted by $\mathcal{F}_{\mathfrak{p}}$ and is called the **reduction modulo** p of $\mathcal{F}$.*

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Suppose that an abstract property $\boldsymbol{P}$ holds for $\mathcal{F}_{\mathfrak{p}}$ for an infinitely many primes (or almost all primes) $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$. What we can say about $\mathcal{F}$?

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- **infinitely many primes** = primes in some dense subset of $\mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$;
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When $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}$ then the notions: **infinitely many primes** and **all most primes** are the usual notions.

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Proposition

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ and suppose that $\mathcal{F}_{\mathfrak{p}}$ has an invariant curve of degree less than $\boldsymbol{d}$ for almost all primes $\mathfrak{p}$. Then, $\mathcal{F}$ has an invariant curve of degree less than $\boldsymbol{d}$.

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Idea: the set $S(\mathcal{F}, K, d)$ of foliations on $\mathbb{P}^2_K$ that have invariant curves of degree $\leq d$ is algebraic variety over K. In particular, $S(\mathcal{F}, \mathbb{C}, d) \neq \varnothing$ if and only if $S(\mathcal{F}, \overline{\mathbb{F}}_{\mathfrak{p}}, d) \neq \varnothing$ for almost all primes $\mathfrak{p}$.

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The p-**divisor** is given by

$$\Delta_{\mathcal{F}} = \{i_{v_\omega^p}\omega = 0\}.$$

Note that $\Delta_{\mathcal{F}}$ has degree $p(d-1) + d + 2$.

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By iteration we get:

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2-\alpha^p}{3}\right) y\partial_y + \left(\frac{-1-\alpha^p}{3}\right) z\partial_z$$

and the equation for the p-divisor is:

$$i_{v^p}\omega = yzv^p(x) - \alpha xzv^p(y) + (\alpha - 1)xyv^p(z) = (\alpha^p - \alpha)xyz$$

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[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

Corollary

On the projective plane over characteristic $p > 0$ any foliation of degree d such that $p \nmid d + 2$ has an invariant algebraic curve.

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[a] *A foliation is p-closed if and only if it has infinitely many solutions.*

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Theorem

[a] *Let $p > 2$ be a prime number such that $7 \nmid p+4$ and $p \not\equiv 1 \mod 3$. Then the Jouanolou foliation, $\mathcal{F}_2$, over characteristic $p > 0$ has an unique invariant algebraic curve and that curve has degree $p+4$.*

[a] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**

Part III: Applications to foliations over $\mathbb{C}$

Algebraic solutions

Goal: use reduction modulo p and property of the p-divivor to prove the non-algebraicity of foliations.

[1] Carnicer - **The Poincare problem in the nondicritical case**

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[a] *Let $\mathcal{F}$ be a non-dicritical foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by a projective 1-form $\omega = Adx + Bdy + Cdz$ with $A, B, C \in \mathbb{Z}[x, y, z]$. Let p be a prime number such that $p > d + 2$. If $\Delta_{\mathcal{F}_p}$ is irreducible then $\mathcal{F}$ has no algebraic solutions.*

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Idea: Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. We can assume that $F \in \mathbb{Z}[x, y, z]$. The Carnicer bound[1] implies that $\deg(C) \leq d + 2$. Reducing modulo p and using the irreducibility of $\Delta_{\mathcal{F}_p}$ we get a contradiction.

[1] Carnicer - **The Poincare problem in the nondicritical case**

Applications

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Idea: Suppose that there is a invariant curve $C = \{F = 0\}$ that is $\mathcal{F}$-invariant. This curve has degree e. For large primes p we obtain $C \mod p = \Delta_{\mathcal{F}_p}$, a contradiction since the degree of the p-divisor depends of p.

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Example

Let Φ be the morphism

$$\Phi \colon \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad [x : y : z] \mapsto [x^2 : y^2 : z^2]$$

and consider $\mathcal{G} = \Phi^* \mathcal{F}_d$, where $\mathcal{F}_d$ is the Jouanolou foliation of degree d. Then, $\mathcal{G}_{\mathfrak{p}}$ is not p-closed for infinitely many primes $\mathfrak{p}$ with the p-divisor having a p-component.

Invariant curves via characteristic 2

Using reduction modulo two it is possible to give a new proof of the following result[2]:

[2] W.Mendson - **Arithmetic aspects of the Jouanolou foliation**
[3] J.V.Pereira, P.F.Sánchez - **Automorphim and non-integrability**

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Theorem

Let $d \in \mathbb{Z}$ such that $d \not\equiv 1 \mod 3$ and $d \equiv 1 \mod 2$. If $K = \mathbb{C}$ then the Jouanolou foliation of degree d has no algebraic solutions.

The idea of the proof:

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step 1: Suppose that there is an algebraic curve C_0 given by an irreducible polynomial $F \in \mathbb{C}[x, y, z]$;

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step 2: An particular property of the automorphim group of $\mathcal{F}_d$[3] permits to construct a invariant curve C_1, from C_0, that has degree $d + 2$ and it is defined over a number field L. In particular, $\deg(C_1) \equiv 1 \mod 2$;

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step 3: The condition $d \equiv 1 \mod 2$ implies that the foliation $\mathcal{F}_d \mod 2\mathbb{Z}$ is not 2-closed. **Main fact:** the 2-divisor is irreducible of degree $3d$;

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step 4: Reducing C_1 modulo 2 we get a contradiction by degree comparison since $C_1 = \Delta_{\mathcal{F}_d} \mod 2$.

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Some comments in higher dimension

The technique of reduction modulo p for codimension one foliation on the projective spaces can be used to give a proof of the following theorem about irreducible components:

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Theorem A

[a][b] *Let $d \in \mathbb{Z}_{\geq 3}$ and $d_1, d_2 \in \mathbb{Z}_{>0}$ such that $d = d_1 + d_2 + 2$. Consider the rational map*

$$\Psi\colon \mathrm{Map}_1(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \times \mathbb{F}\mathrm{ol}_{(\mathrm{d}_1,\mathrm{d}_2)}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \dashrightarrow \mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$$
$$(\Phi, \mathcal{G}) \mapsto \Phi^*\mathcal{G}.$$

Let U be the open set of definition of Ψ and $C_{(d;d_1,d_2)}$ the Zariski closure of $\Psi(U)$. Then, $C_{(d;d_1,d_2)}$ is an irreductible component of $\mathbb{F}\mathrm{ol}_{\mathrm{d}}(\mathbb{P}^3_{\mathbb{C}})$.

[a] W.Mendson - **Folheações de codimensão um em característica positiva e aplicações**
[b] W.Mendson, J.V.Pereira - **Codimension one foliations in positive characteristic**

The topics in the proof include:

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- proving the analogous theorem in positive characteristic and lift to characteristic 0.

Thank you ;-)